Contents 4

Trigonometry

Learning outcomes

In this Workbook you will learn about the basic building blocks of trigonometry. You will learn about the sine, cosine, tangent, cosecant, secant, cotangent functions and their many important relationships. You will learn about their graphs and their periodic nature. You will learn how to apply Pythagoras' theorem and the Sine and Cosine rules to find lengths and angles of triangles.

Right-angled Triangles

4.1

Introduction

Right-angled triangles (that is triangles where one of the angles is 90°) are the easiest topic for introducing trigonometry. Since the sum of the three angles in a triangle is 180° it follows that in a right-angled triangle there are no obtuse angles (i.e. angles greater than 90°). In this Section we study many of the properties associated with right-angled triangles.

Prerequisites

Before starting this Section you should ...

- have a basic knowledge of the geometry of triangles

Learning Outcomes

On completion you should be able to ...

- define trigonometric functions both in right-angled triangles and more generally
- express angles in degrees
- calculate all the angles and sides in any right-angled triangle given certain information

1. Right-angled triangles

Look at Figure 1 which could, for example, be a profile of a hill with a constant gradient.

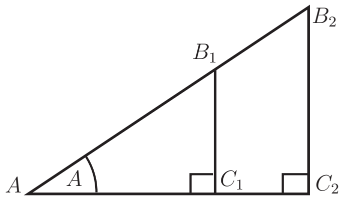


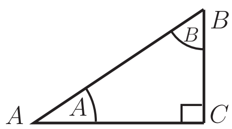
Figure 1

The two right-angled triangles AB_1C_1 and AB_2C_2 are **similar** (because the three angles of triangle AB_1C_1 are equal to the equivalent 3 angles of triangle AB_2C_2). From the basic properties of similar triangles corresponding sides have the same ratio. Thus, for example,

$$\frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2} \qquad \text{and} \qquad \frac{AC_1}{AB_1} = \frac{AC_2}{AB_2} \qquad (1)$$

The values of the two ratios (1) will clearly depend on the angle A of inclination. These ratios are called the **sine** and **cosine** of the angle A, these being abbreviated to $\sin A$ and $\cos A$.

Key Point 1



$$\sin A = \frac{BC}{AB} \qquad \cos A = \frac{AC}{AB}$$

Figure 2

AC is the side **adjacent** to angle A.

BC is the side **opposite** to angle A.

AB is the **hypotenuse** of the triangle (the longest side).

Task

Referring again to Figure 2 in Key Point 1, write down the ratios which give $\sin B$ and $\cos B$.

Your solution

Answer

$\sin B = \dfrac{AC}{AB} \qquad \cos B = \dfrac{BC}{AB}$.

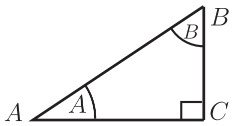
Note that $\sin B = \cos A = \cos(90° - B)$ and $\cos B = \sin A = \sin(90° - B)$

A third result of importance from Figure 1 is

$$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} \qquad (2)$$

These ratios is referred to as the **tangent** of the angle at A, written $\tan A$.

Key Point 2



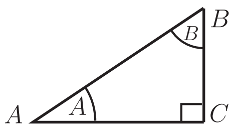
$$\sin A = \frac{BC}{AB} \qquad \cos A = \frac{AC}{AB}$$

Figure 3

$$\tan A = \frac{BC}{AC} = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

For any right-angled triangle the values of sine, cosine and tangent are given in Key Point 3.

Key Point 3



$$\sin A = \frac{BC}{AB} \qquad \cos A = \frac{AC}{AB}$$

Figure 4

We can write, therefore, for any right-angled triangle containing an angle θ (not the right-angle)

$$\begin{aligned}
\sin\theta &= \frac{\text{length of side opposite angle } \theta}{\text{length of hypotenuse}} = \frac{\text{Opp}}{\text{Hyp}} \\
\cos\theta &= \frac{\text{length of side adjacent to angle } \theta}{\text{length of hypotenuse}} = \frac{\text{Adj}}{\text{Hyp}} \\
\tan\theta &= \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta} = \frac{\text{Opp}}{\text{Adj}}
\end{aligned}$$

These are sometimes memorised as SOH, CAH and TOA respectively.

These three ratios are called **trigonometric ratios**.

Task

Write $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Your solution

Answer

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}} = \frac{\text{Opp}}{\text{Adj}}.\frac{\text{Hyp}}{\text{Hyp}} = \frac{\text{Opp}}{\text{Hyp}}.\frac{\text{Hyp}}{\text{Adj}} = \frac{\text{Opp}}{\text{Hyp}}\Big/\frac{\text{Adj}}{\text{Hyp}} \qquad \text{i.e. } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Key Point 4

Pythagoras' Theorem

$$a^2 + b^2 = c^2$$

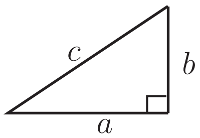


Figure 5

Example 1

Use the isosceles triangle in Figure 6 to obtain the sine, cosine and tangent of $45°$.

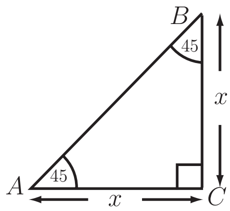


Figure 6

Solution

By Pythagoras' theorem $\quad (AB)^2 = x^2 + x^2 = 2x^2 \quad$ so $\quad AB = x\sqrt{2}$

Hence $\quad \sin 45° = \dfrac{BC}{AB} = \dfrac{x}{x\sqrt{2}} = \dfrac{1}{\sqrt{2}} \qquad \cos 45° = \dfrac{AC}{AB} = \dfrac{1}{\sqrt{2}} \qquad \tan 45° = \dfrac{BC}{AC} = \dfrac{x}{x} = 1$

Engineering Example 1

Noise reduction by sound barriers

Introduction

Audible sound has much longer wavelengths than light. Consequently, sound travelling in the atmosphere is able to bend around obstacles even when these obstacles cause sharp shadows for light. This is the result of the wave phenomenon known as **diffraction**. It can be observed also with water waves at the ends of breakwaters. The extent to which waves bend around obstacles depends upon the wavelength and the source-receiver geometry. So the efficacy of purpose built noise barriers, such as to be found alongside motorways in urban and suburban areas, depends on the frequencies in the sound and the locations of the source and receiver (nearest noise-affected person or dwelling) relative to the barrier. Specifically, the barrier performance depends on the difference in the lengths of the hypothetical ray paths passing from source to receiver either directly or via the top of the barrier (see Figure 7).

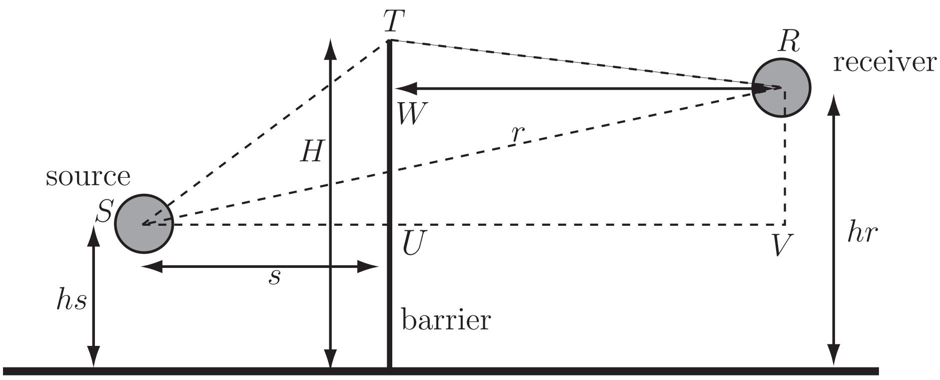


Figure 7

Problem in words

Find the difference in the path lengths from source to receiver either directly or via the top of the barrier in terms of

(i) the source and receiver heights,
(ii) the horizontal distances from source and receiver to the barrier and
(iii) the height of the barrier.

Calculate the path length difference for a 1 m high source, 3 m from a 3 m high barrier when the receiver is 30 m on the other side of the barrier and at a height of 1 m.

Mathematical statement of the problem

Find $ST + TR - SR$ in terms of hs, hr, s, r and H.

Calculate this quantity for $hs = 1,\ s = 3,\ H = 3,\ r = 30$ and $hr = 1$.

Mathematical analysis

Note the labels V, U, W on points that are useful for the analysis. Note that the length of $RV = hr - hs$ and that the horizontal separation between S and R is $r + s$. In the right-angled triangle SRV, Pythagoras' theorem gives

$$(SR)^2 = (r+s)^2 + (hr - hs)^2$$

So

$$SR = \sqrt{(r+s)^2 + (hr-hs)^2} \qquad (3)$$

Note that the length of $TU = H - hs$ and the length of $TW = H - hr$. In the right-angled triangle STU,

$$(ST)^2 = s^2 + (H - hs)^2$$

In the right-angled triangle TWR,

$$(TR)^2 = r^2 + (H - hr)^2$$

So

$$ST + TR = \sqrt{s^2 + (H-hs)^2} + \sqrt{r^2 + (H-hr)^2} \qquad (4)$$

So using (3) and (4)

$$ST + TR - SR = \sqrt{s^2 + (H-hs)^2} + \sqrt{r^2 + (H-hr)^2} - \sqrt{(r+s)^2 + (hr-hs)^2}.$$

For $hs = 1,\ s = 3,\ H = 3,\ r = 30$ and $hr = 1$,

$$\begin{aligned} ST + TR - SR &= \sqrt{3^2 + (3-1)^2} + \sqrt{30^2 + (3-1)^2} - \sqrt{(30+3)^2 + (1-1)^2} \\ &= \sqrt{13} + \sqrt{904} - 33 \\ &= 0.672 \end{aligned}$$

So the path length difference is 0.672 m.

Interpretation

Note that, for equal source and receiver heights, the further either receiver or source is from the barrier, the smaller the path length difference. Moreover if source and receiver are at the same height as the barrier, the path length difference is zero. In fact diffraction by the barrier still gives some sound reduction for this case. The smaller the path length difference, the more accurately it has to be calculated as part of predicting the barriers noise reduction.

Engineering Example 2

Horizon distance

Problem in words

Looking from a height of 2 m above sea level, how far away is the horizon? State any assumptions made.

Mathematical statement of the problem

Assume that the Earth is a sphere. Find the length D of the tangent to the Earth's sphere from the observation point O.

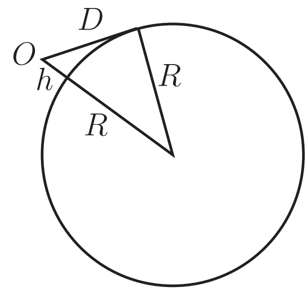


Figure 8: The Earth's sphere and the tangent from the observation point O

Mathematical analysis

Using Pythagoras' theorem in the triangle shown in Figure 8,

$$(R+h)^2 = D^2 + R^2$$

Hence

$$R^2 + 2Rh + h^2 = D^2 + R^2 \quad \rightarrow \quad h(2R+h) = D^2 \quad \rightarrow \quad D = \sqrt{h(2R+h)}$$

If $R = 6.373 \times 10^6$ m, then the variation of D with h is shown in Figure 9.

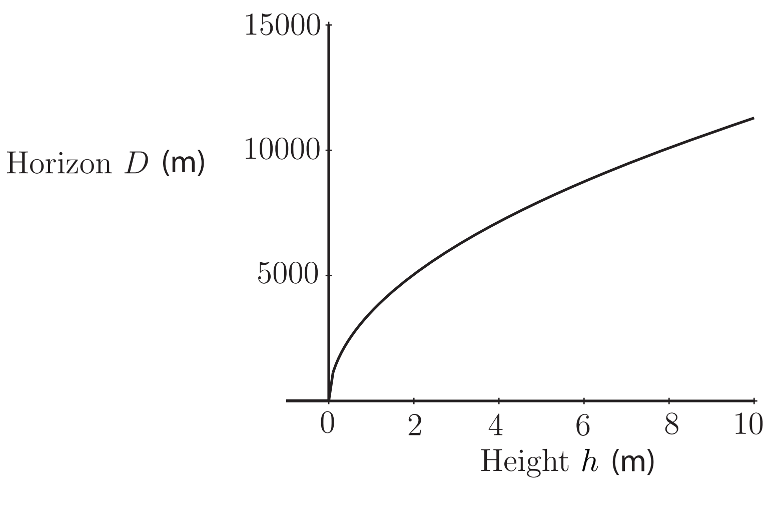


Figure 9

At an observation height of 2 m, the formula predicts that the horizon is just over 5 km away. In fact the variation of optical refractive index with height in the atmosphere means that the horizon is approximately 9% greater than this.

Task

Using the triangle ABC in Figure 10 which can be regarded as one half of the equilateral triangle ABD, calculate sin, cos, tan for the angles $30°$ and $60°$.

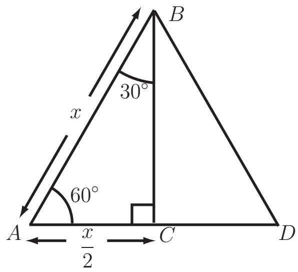


Figure 10

Your solution

Answer

By Pythagoras' theorem: $(BC)^2 = (AB)^2 - (AC)^2 = x^2 - \dfrac{x^2}{4} = \dfrac{3x^2}{4}$ so $BC = x\dfrac{\sqrt{3}}{2}$

Hence $\sin 60° = \dfrac{BC}{AB} = \dfrac{x\frac{\sqrt{3}}{2}}{x} = \dfrac{\sqrt{3}}{2}$ $\quad \sin 30° = \dfrac{AC}{AB} = \dfrac{\frac{x}{2}}{x} = \dfrac{1}{2}$ $\quad \cos 60° = \dfrac{AC}{AB} = \dfrac{1}{2}$

$\cos 30° = \dfrac{BC}{AB} = \dfrac{x\frac{\sqrt{3}}{2}}{x} = \dfrac{\sqrt{3}}{2}$ $\quad \tan 60° = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$ $\quad \tan 30° = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$

Values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can of course be obtained by calculator. When entering the angle in degrees (e.g. $30°$) the calculator must be in degree mode. (Typically this is ensured by pressing the DRG button until 'DEG' is shown on the display). The keystrokes for $\sin 30°$ are usually simply $\boxed{\sin}$ $\boxed{30}$ or, on some calculators, $\boxed{30}$ $\boxed{\sin}$ perhaps followed by $\boxed{=}$.

Task

(a) Use your calculator to check the values of $\sin 45°$, $\cos 30°$ and $\tan 60°$ obtained in the previous Task.

(b) Also obtain $\sin 3.2°$, $\cos 86.8°$, $\tan 28°15'$. ($'$ denotes a minute $= \dfrac{1}{60}^{\circ}$)

Your solution
(a)
(b)

Answer

(a) $0.7071,\ 0.8660,\quad 1.7321$ to 4 d.p.

(b) $\sin 3.2° = \cos 86.8° = 0.0558$ to 4 d.p., $\quad \tan 28°15' = \tan 28.25° = 0.5373$ to 4 d.p.

Inverse trigonometric functions (a first look)

Consider, by way of example, a right-angled triangle with sides 3, 4 and 5, see Figure 11.

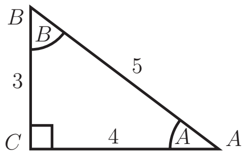


Figure 11

Suppose we wish to find the angles at A and B. Clearly $\sin A = \dfrac{3}{5}$, $\cos A = \dfrac{4}{5}$, $\tan A = \dfrac{3}{4}$ so we need to solve one of the above three equations to find A.

Using $\sin A = \dfrac{3}{5}$ we write $\quad A = \sin^{-1}\left(\dfrac{3}{5}\right)\quad$ (read as 'A is the inverse sine of $\dfrac{3}{5}$')

The value of A can be obtained by calculator using the '$\sin^{-1}$' button (often a second function to the $\sin$ function and accessed using a $\boxed{\text{SHIFT}}$ or $\boxed{\text{INV}}$ or $\boxed{\text{SECOND FUNCTION}}$ key).

Thus to obtain $\sin^{-1}\left(\dfrac{3}{5}\right)$ we might use the following keystrokes:

$$\boxed{\text{INV}}\ \boxed{\text{SIN}}\ \boxed{0.6}\ \boxed{=}\quad \text{or}\quad \boxed{3} \div \boxed{5}\ \boxed{\text{INV}}\ \boxed{\text{SIN}}\ \boxed{=}$$

We find $\sin^{-1}\dfrac{3}{5} = 36.87°$ (to 4 significant figures).

Key Point 5

Inverse Trigonometric Functions

$$\sin\theta = x \quad \text{implies} \quad \theta = \sin^{-1} x$$
$$\cos\theta = y \quad \text{implies} \quad \theta = \cos^{-1} y$$
$$\tan\theta = z \quad \text{implies} \quad \theta = \tan^{-1} z$$

(The alternative notations arc$\sin$, arc$\cos$, arc$\tan$ are sometimes used for these inverse functions.)

Task

Check the values of the angles at A and B in Figure 11 above using the $\cos^{-1}$ functions on your calculator. Give your answers in degrees to 2 d.p.

Your solution

Answer

$A = \cos^{-1}\dfrac{4}{5} = 36.87°$ $\qquad B = \cos^{-1}\dfrac{3}{5} = 53.13°$

Task

Check the values of the angles at A and B in Figure 11 above using the $\tan^{-1}$ functions on your calculator. Give your answers in degrees to 2 d.p.

Your solution

Answer

$A = \tan^{-1}\dfrac{3}{4} = 36.87°$ $\qquad B = \tan^{-1}\dfrac{4}{3} = 53.13°$

You should note carefully that $\sin^{-1} x$ **does not mean** $\dfrac{1}{\sin x}$.

Indeed the function $\dfrac{1}{\sin x}$ has a special name – the cosecant of x, written $\text{cosec}\, x$. So

$$\text{cosec}\, x \equiv \frac{1}{\sin x} \quad \text{(the \textbf{cosecant} function).}$$

Similarly

$$\sec x \equiv \frac{1}{\cos x} \quad \text{(the \textbf{secant} function)}$$

$$\cot x \equiv \frac{1}{\tan x} \quad \text{(the \textbf{cotangent} function).}$$

Use your calculator to obtain to 3 d.p. $\text{cosec}\ 38.5°$, $\sec 22.6°$, $\cot 88.32°$ (Use the sin, cos or tan buttons unless your calculator has specific buttons.)

Your solution

Answer

$$\text{cosec}\ 38.5° = \frac{1}{\sin 38.5°} = 1.606 \qquad \sec 22.6° = \frac{1}{\cos 22.6°} = 1.083$$

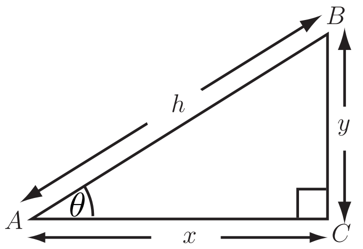
$$\cot 88.32° = \frac{1}{\tan 88.32°} = 0.029$$

2. Solving right-angled triangles

Solving right-angled triangles means obtaining the values of all the angles and all the sides of a given right-angled triangle using the trigonometric functions (and, if necessary, the inverse trigonometric functions) and perhaps Pythagoras' theorem.
There are three cases to be considered:

Case 1 Given the hypotenuse and an angle

We use $\sin$ or cos as appropriate:



(a)

Figure 12

Assuming h and θ in Figure 12 are given then

$$\cos\theta = \frac{x}{h} \text{ which gives } x = h\cos\theta$$

from which x can be calculated.

Also

$$\sin\theta = \frac{y}{h} \text{ so } y = h\sin\theta \text{ which enables us to calculate } y.$$

Clearly the third angle of this triangle (at B) is $90° - \theta$.

Case 2 Given a side other than the hypotenuse and an angle.

We use $\tan$:

(a) If x and θ are known then, in Figure 12, $\tan\theta = \dfrac{y}{x}$ so $y = x\tan\theta$

which enables us to calculate y.

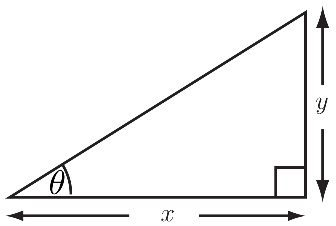
(b) If y and θ are known then $\tan\theta = \dfrac{y}{x}$ gives $x = \dfrac{y}{\tan\theta}$ from which x can be calculated.

Then the hypotenuse can be calculated using Pythagoras' theorem: $h = \sqrt{x^2 + y^2}$

Case 3 Given two of the sides

We use $\tan^{-1}$ or $\sin^{-1}$ or $\cos^{-1}$:

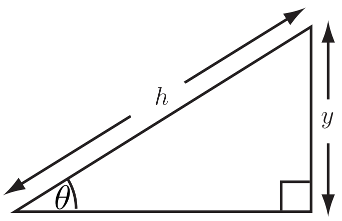
(a)



$$\tan\theta = \frac{y}{x} \qquad \text{so} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Figure 13

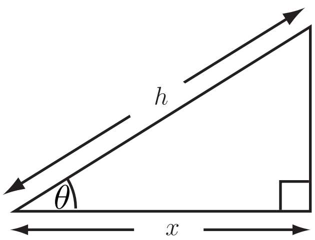
(b)



$$\sin\theta = \frac{y}{h} \qquad \text{so} \qquad \theta = \sin^{-1}\left(\frac{y}{h}\right)$$

Figure 14

(c)



$$\cos\theta = \frac{x}{h} \qquad \text{so} \qquad \theta = \cos^{-1}\left(\frac{x}{h}\right)$$

Figure 15

Note: since two sides are given we can use Pythagoras' theorem to obtain the length of the third side at the outset.

Engineering Example 3

Vintage car brake pedal mechanism

Introduction

Figure 16 shows the structure and some dimensions of a vintage car brake pedal arrangement as far as the brake cable. The **moment** of a force about a point is the product of the force and the perpendicular distance from the point to the line of action of the force. The pedal is pivoted about the point A. The moments about A must be equal as the pedal is stationary.

Problem in words

If the driver supplies a force of 900 N, to act at point B, calculate the force (F) in the cable.

Mathematical statement of problem

The perpendicular distance from the line of action of the force provided by the driver to the pivot point A is denoted by x_1 and the perpendicular distance from the line of action of force in the cable to the pivot point A is denoted by x_2. Use trigonometry to relate x_1 and x_2 to the given dimensions. Calculate clockwise and anticlockwise moments about the pivot and set them equal.

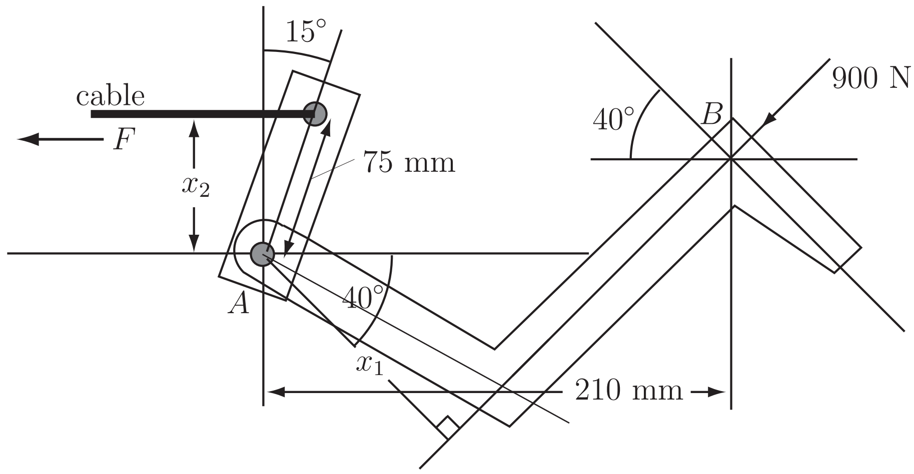
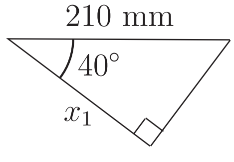


Figure 16: Structure and dimensions of vintage car brake pedal arrangement

Mathematical Analysis

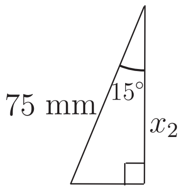
The distance x_1 is found by considering the right-angled triangle shown in Figure 17 and using the definition of cosine.



$$\cos(40^\circ) = \frac{x_1}{0.210} \qquad \text{hence} \qquad x_1 = 161 \text{ mm.}$$

Figure 17

The distance x_2 is found by considering the right-angled triangle shown in Figure 18.



$$\cos(15°) = \frac{x_2}{0.075} \qquad \text{hence} \qquad x_2 = 72 \text{ mm.}$$

Figure 18

Equating moments about A:

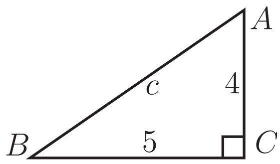
$900x_1 = Fx_2$ so $F = 2013$ N.

Interpretation

This means that the force exerted by the cable is 2013 N in the direction of the cable. This force is more than twice that applied by the driver. In fact, whatever the force applied at the pedal the force in the cable will be more than twice that force. The pedal structure is an example of a lever system that offers a mechanical gain.

Task

Obtain all the angles and the remaining side for the triangle shown:



Your solution

Answer

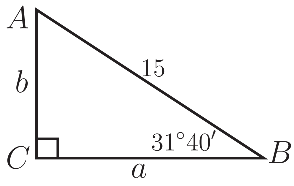
This is Case 3. To obtain the angle at B we use $\tan B = \dfrac{4}{5}$ so $B = \tan^{-1}(0.8) = 38.66°$.

Then the angle at A is $180° - (90° - 38.66°) = 51.34°$.

By Pythagoras' theorem $c = \sqrt{4^2 + 5^2} = \sqrt{41} \approx 6.40$.

Task

Obtain the remaining sides and angles for the triangle shown.



Your solution

Answer

This is Case 1. Since $31^\circ 40' = 31.67^\circ$ then $\cos 31.67^\circ = \dfrac{a}{15}$ so $a = 15\cos 31.67^\circ = 12.77.$

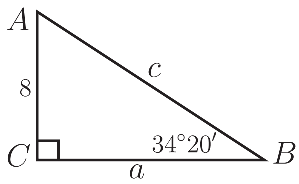
The angle at A is $180^\circ - (90 + 31.67^\circ) = 58.33^\circ.$

Finally $\sin 31.67^\circ = \dfrac{b}{15}$ $\therefore$ $b = 15\sin 31.67^\circ = 7.85.$

(Alternatively, of course, Pythagoras' theorem could be used to calculate the length b.)

Task

Obtain the remaining sides and angles of the following triangle.



Your solution

Answer

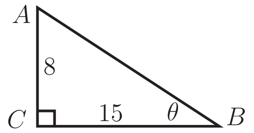
This is Case 2.

Here $\tan 34.33^\circ = \dfrac{8}{a}$ so $a = \dfrac{8}{\tan 34.33^\circ} = 11.7$

Also $c = \sqrt{8^2 + 11.7^2} = 14.18$ and the angle at A is $180^\circ - (90^\circ + 34.33^\circ) = 55.67^\circ.$

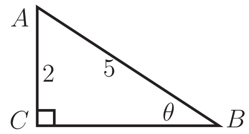
Exercises

1. Obtain $\operatorname{cosec}\theta$, $\sec\theta$, $\cot\theta$, θ in the following right-angled triangle.

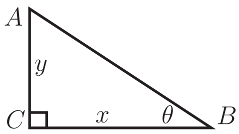


2. Write down $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$ for each of the following triangles:

(a)



(b)

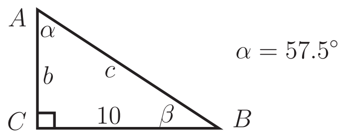


3. If θ is an acute angle such that $\sin\theta = 2/7$ obtain, without use of a calculator, $\cos\theta$ and $\tan\theta$.

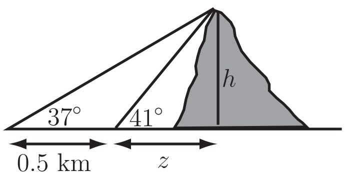
4. Use your calculator to obtain the acute angles θ satisfying

(a) $\sin\theta = 0.5260$, (b) $\tan\theta = 2.4$, (c) $\cos\theta = 0.2$

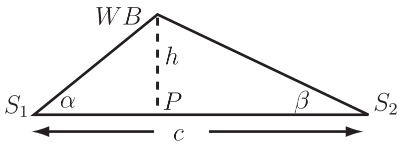
5. Solve the right-angled triangle shown:



6. A surveyor measures the **angle of elevation** between the top of a mountain and ground level at two different points. The results are shown in the following figure. Use trigonometry to obtain the distance z (which cannot be measured) and then obtain the height h of the mountain.

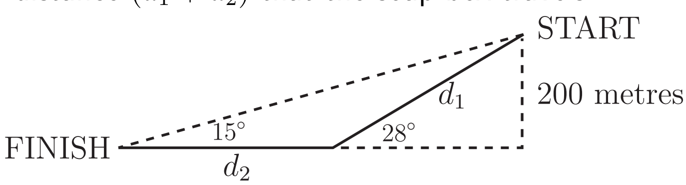


7. As shown below two tracking stations S_1 and S_2 sight a weather balloon (WB) between them at elevation angles α and β respectively.



Show that the height h of the balloon is given by $h = \dfrac{c}{\cot\alpha + \cot\beta}$

8. A vehicle entered in a 'soap box derby' rolls down a hill as shown in the figure. Find the total distance $(d_1 + d_2)$ that the soap box travels.



Answers

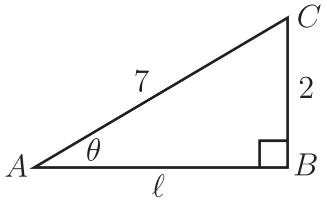
1. $h = \sqrt{15^2+8^2} = 17, \quad \text{cosec}\,\theta = \dfrac{1}{\sin\theta} = \dfrac{17}{8} \quad \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{17}{15} \qquad \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{15}{8}$

   $\theta = \sin^{-1}\dfrac{8}{17}$ (for example) $\quad \therefore \quad \theta = 28.07^\circ$

2. (a) $\sin\theta = \dfrac{2}{5} \quad \cos\theta = \dfrac{\sqrt{21}}{5} \quad \tan\theta = \dfrac{2\sqrt{21}}{21} \quad \text{cosec}\,\theta = \dfrac{5}{2}$

   (b) $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}} \quad \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}} \quad \tan\theta = \dfrac{y}{x} \quad \text{cosec}\,\theta = \dfrac{\sqrt{x^2+y^2}}{y}$

3. Referring to the following diagram



   $\ell = \sqrt{7^2-2^2} = \sqrt{45} = 3\sqrt{5}$

   Hence $\cos\theta = \dfrac{3\sqrt{5}}{7} \quad \tan\theta = \dfrac{2}{3\sqrt{5}} = \dfrac{2\sqrt{5}}{15}$

4. (a) $\theta = \sin^{-1}0.5260 = 31.73^\circ$ (b) $\theta = \tan^{-1}2.4 = 67.38^\circ$ (c) $\theta = \cos^{-1}0.2 = 78.46^\circ$

5. $\beta = 90-\alpha = 32.5^\circ, \qquad b = \dfrac{10}{\tan 57.5^\circ} \simeq 6.37 \qquad c = \dfrac{10}{\sin 57.5^\circ} \simeq 11.86$

6. $\tan 37^\circ = \dfrac{h}{z+0.5} \qquad \tan 41^\circ = \dfrac{h}{z}$ from which

   $h = (z+0.5)\tan 37^\circ = z\tan 41^\circ$, so $z\tan 37^\circ - z\tan 41^\circ = -0.5\tan 37^\circ$

   $\therefore \quad z = \dfrac{-0.5\tan 37^\circ}{\tan 37^\circ - \tan 41^\circ} \simeq 3.2556$ km, so $h = z\tan 41^\circ = 3.2556\tan 41^\circ \simeq 2.83$ km

7. Since the required answer is in terms of $\cot\alpha$ and $\cot\beta$ we proceed as follows:

   Using x to denote the distance $S_1P \quad \cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{x}{h} \qquad \cot\beta = \dfrac{1}{\tan\beta} = \dfrac{c-x}{h}$

   Adding: $\cot\alpha + \cot\beta = \dfrac{x}{h} + \dfrac{c-x}{h} = \dfrac{c}{h} \qquad \therefore \quad h = \dfrac{c}{\cot\alpha+\cot\beta}$ as required.

8. From the smaller right-angled triangle $d_1 = \dfrac{200}{\sin 28^\circ} = 426.0$ m. The base of this triangle then has length $\ell = 426\cos 28^\circ = 376.1$ m

   From the larger right-angled triangle the straight-line distance from START to FINISH is $\dfrac{200}{\sin 15^\circ} = 772.7$ m. Then, using Pythagoras' theorem $(d_2+\ell) = \sqrt{772.7^2-200^2} = 746.4$ m from which $d_2 = 370.3$ m $\therefore \quad d_1+d_2 = 796.3$ m

Trigonometric Functions

4.2

Introduction

Our discussion so far has been limited to right-angled triangles where, apart from the right-angle itself, all angles are necessarily less than $90°$. We now extend the definitions of the trigonometric functions to any size of angle, which greatly broadens the range of applications of trigonometry.

Prerequisites

Before starting this Section you should ...

- have a basic knowledge of the geometry of triangles

Learning Outcomes

On completion you should be able to ...

- express angles in radians
- define trigonometric functions generally
- sketch the graphs of the three main trigonometric functions: $\sin, \cos, \tan$

1. Trigonometric functions for any size angle

The radian

First we introduce an alternative to measuring angles in degrees. Look at the circle shown in Figure 19(a). It has radius r and we have shown an arc AB of length ℓ (measured in the same units as r.) As you can see the arc subtends an angle θ at the centre O of the circle.

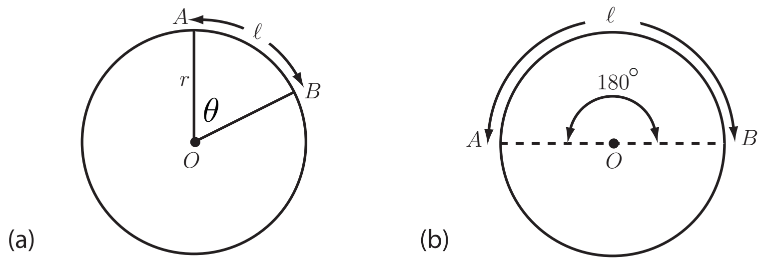


Figure 19

The angle θ in **radians** is defined as

$$\theta = \frac{\text{length of arc } AB}{\text{radius}} = \frac{\ell}{r}$$

So, for example, if $r = 10$ cm, $\ell = 20$ cm, the angle θ would be $\dfrac{20}{10} = 2$ radians.

The relation between the value of an angle in radians and its value in degrees is readily obtained as follows. Referring to Figure 19(b) imagine that the arc AB extends to cover half the complete perimeter of the circle. The arc length is now πr (half the circumference of the circle) so the angle θ subtended by AB is now

$$\theta = \frac{\pi r}{r} = \pi \quad \text{radians}$$

But clearly this angle is $180°$. Thus π radians is the same as $180°$.

Note conversely that since π radians $= 180°$ then 1 radian $= \dfrac{180}{\pi}$ degrees (about $57.3°$).

Key Point 6

$180° = \pi$ radians

$360° = 2\pi$ radians

$1° = \dfrac{\pi}{180}$ radians

$x° = \dfrac{\pi x}{180}$ radians

1 radian $= \dfrac{180}{\pi}$ degrees ($\approx 57.3°$)

y radians $= \dfrac{180y}{\pi}$ degrees

Task

Write down the values in radians of $30°,\ 45°,\ 90°,\ 135°$. (Leave your answers as multiples of π.)

Your solution

Answer

$$30° = \pi \times \frac{30}{180} = \frac{\pi}{6} \text{ radians} \qquad 45° = \frac{\pi}{4} \text{ radians} \qquad 90° = \frac{\pi}{2} \text{ radians} \qquad 135° = \frac{3\pi}{4} \text{ radians}$$

Task

Write in degrees the following angles given in radians

$$\frac{\pi}{10}, \quad \frac{\pi}{5}, \quad \frac{7\pi}{10}, \quad \frac{23\pi}{12}$$

Your solution

Answer

$$\frac{\pi}{10} \text{ rad } = \frac{180}{\pi} \times \frac{\pi}{10} = 18° \qquad \frac{\pi}{5} \text{ rad } = \frac{180}{\pi} \times \frac{\pi}{5} = 36° \qquad \frac{7\pi}{10} \text{ rad } = \frac{180}{\pi} \times \frac{7\pi}{10} = 126°$$

$$\frac{23\pi}{12} \text{ rad } = \frac{180}{\pi} \times \frac{23\pi}{12} = 345°$$

Task

Put your calculator into **radian mode** (using the DRG button if necessary) for this Task: Verify these facts by first converting the angles to radians:

$$\sin 30° = \frac{1}{2} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 60° = \sqrt{3}$$ (Use the π button to obtain π.)

Your solution

Answer

$$\sin 30° = \sin\left(\frac{\pi}{6}\right) = 0.5, \qquad \cos 45° = \cos\left(\frac{\pi}{4}\right) = 0.7071 = \frac{1}{\sqrt{2}},$$

$$\tan 60° = \tan\left(\frac{\pi}{3}\right) = 1.7320 = \sqrt{3}$$

2. General definitions of trigonometric functions

We now define the trigonometric functions in a more general way than in terms of ratios of sides of a right-angled triangle. To do this we consider a circle of **unit radius** whose centre is at the origin of a Cartesian coordinate system and an arrow (or **radius vector**) OP from the centre to a point P on the circumference of this circle. We are interested in the angle θ that the arrow makes with the **positive** x-axis. See Figure 20.

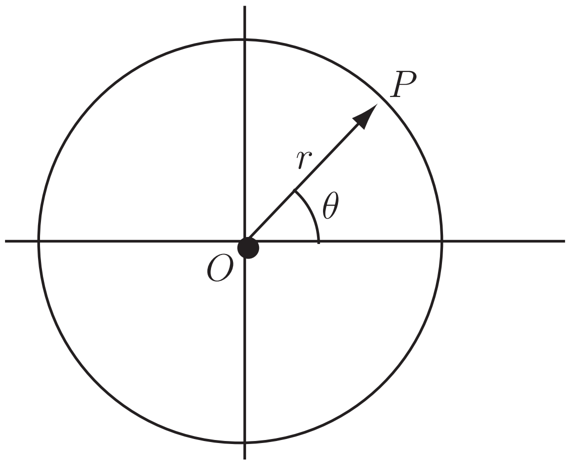
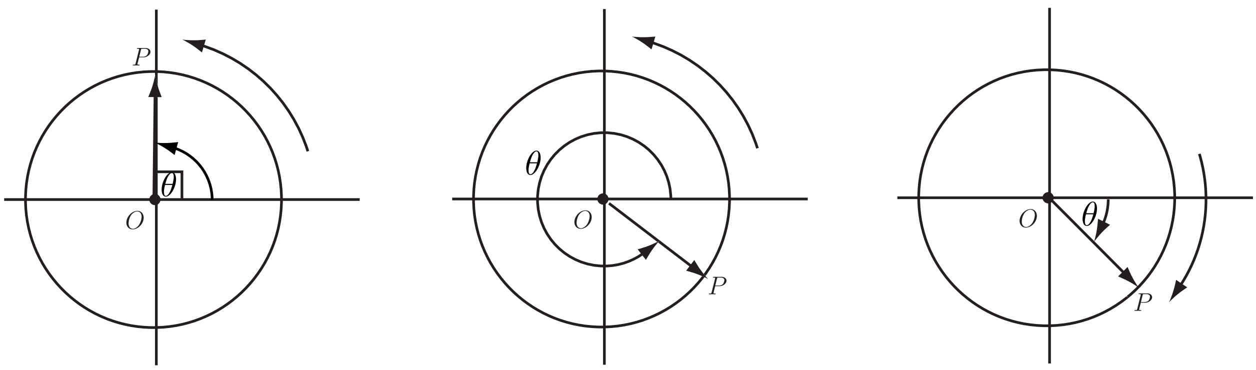


Figure 20

Imagine that the vector OP rotates in **anti-clockwise direction**. With this sense of rotation the angle θ is taken as positive whereas a **clockwise** rotation is taken as negative. See examples in Figure 21.



$\theta = 90^\circ = \dfrac{\pi}{2}$ rad $\qquad \theta = 315^\circ = \dfrac{7\pi}{4}$ rad $\qquad \theta = -45^\circ = -\dfrac{\pi}{4}$ rad

Figure 21

The sine and cosine of an angle

For $0 \leq \theta \leq \dfrac{\pi}{2}$ (called the **first** quadrant) we have the following situation with our unit radius circle. See Figure 22.

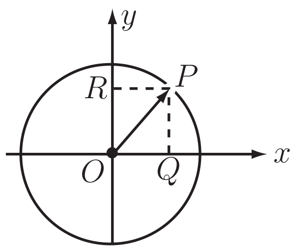


Figure 22

The **projection** of OP along the positive $x-$axis is OQ. But, in the right-angled triangle OPQ

$$\cos\theta = \frac{OQ}{OP} \qquad \text{or} \qquad OQ = OP\cos\theta$$

and since OP has unit length $\qquad \cos\theta = OQ \qquad (3)$

Similarly in this right-angled triangle

$$\sin\theta = \frac{PQ}{OP} \qquad \text{or} \qquad PQ = OP\sin\theta$$

but $PQ = OR$ and OP has unit length

so $\qquad \sin\theta = OR \qquad (4)$

Equation (3) tells us that we can interpret $\cos\theta$ as the projection of OP along the **positive x-axis** and $\sin\theta$ as the projection of OP along the **positive y-axis**.

We shall use these interpretations as the **definitions** of $\sin\theta$ and $\cos\theta$ for **any** values of θ.

Key Point 7

For a radius vector OP of a circle of unit radius making an angle θ with the positive $x-$axis

$\cos\theta =$ projection of OP along the positive $x-$axis

$\sin\theta =$ projection of OP along the positive $y-$axis

Sine and cosine in the four quadrants

First quadrant $(0 \leq \theta \leq 90°)$

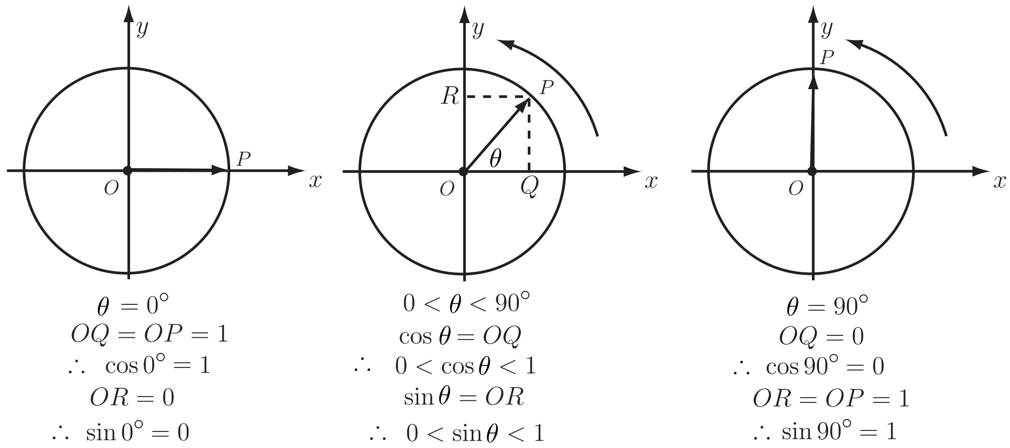


Figure 23

It follows from Figure 23 that $\cos\theta$ decreases from 1 to 0 as OP rotates from the horizontal position to the vertical, i.e. as θ increases from $0°$ to $90°$.

$\sin\theta = OR$ increases from 0 (when $\theta = 0$) to 1 (when $\theta = 90°$).

Second quadrant $(90° \leq \theta \leq 180°)$

Referring to Figure 24, remember that it is the projections along the **positive** x and y axes that are used to define $\cos\theta$ and $\sin\theta$ respectively. It follows that as θ increases from $90°$ to $180°$, $\cos\theta$ decreases from 0 to -1 and $\sin\theta$ decreases from 1 to 0.

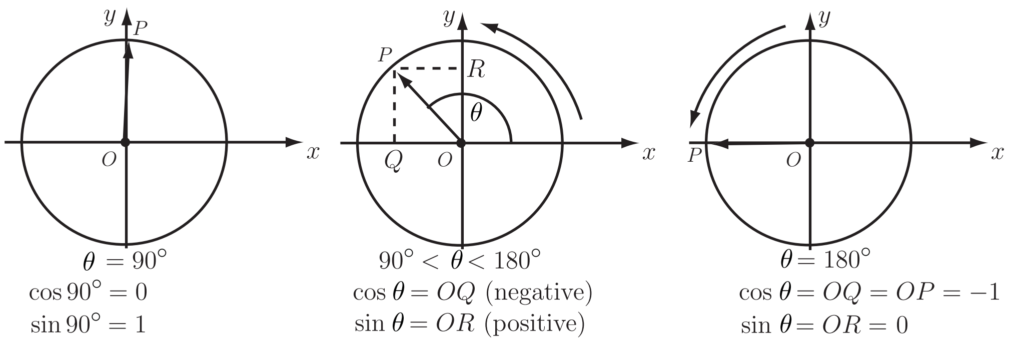


Figure 24

Considering for example an angle of 135°, referring to Figure 25, by symmetry we have:

$$\sin 135° = OR = \sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 135° = OQ_2 = -OQ_1 = -\cos 45° = -\frac{1}{\sqrt{2}}$$

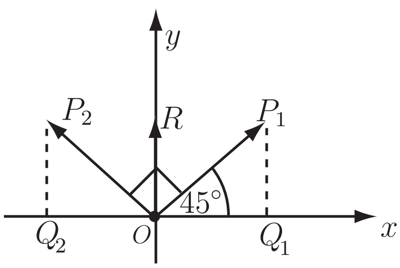


Figure 25

Key Point 8

$\sin(180 - x) \equiv \sin x$ and $\cos(180 - x) \equiv -\cos x$

Task

Without using a calculator write down the values of

$$\sin 120°,\ \sin 150°,\ \cos 120°,\ \cos 150°,\ \tan 120°,\ \tan 150°.$$

(Note that $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ for any value of θ.)

Your solution

Answer

$$\sin 120° = \sin(180 - 60) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\sin 150° = \sin(180 - 30) = \sin 30° = \frac{1}{2}$$

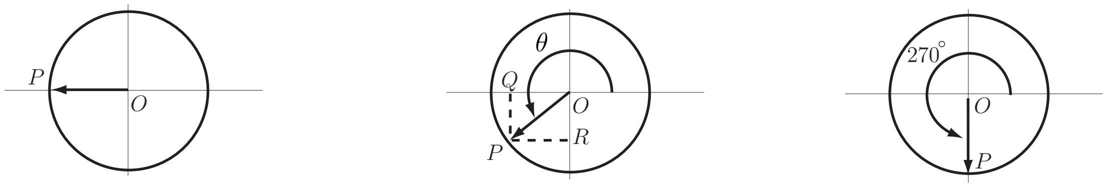
$$\cos 120° = -\cos 60 = -\frac{1}{2}$$

$$\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$$

$$\tan 120° = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

$$\tan 150° = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

Third quadrant $(180° \leq \theta \leq 270°)$.



$\cos 180° = -1$	$180° < \theta < 270°$	$\theta = 270°$
$\sin 180° = 0$	$\cos\theta = OQ$ (negative)	$\cos\theta = ?$
	$\sin\theta = OR$ (negative)	$\sin\theta = ?$

Figure 26

Task

Using the projection definition write down the values of $\cos 270°$ and $\sin 270°$.

Your solution

Answer

$\cos 270° = 0$ (OP has zero projection along the positive $x-$axis)

$\sin 270° = -1$ (OP is directed along the negative axis)

Thus in the third quadrant, as θ increases from $180°$ to $270°$ so $\cos\theta$ increases from -1 to 0 whereas $\sin\theta$ decreases from 0 to -1.

From the results of the last Task, with $\theta = 180° + x$ (see Figure 27) we obtain for all x the relations:

$$\sin\theta = \sin(180 + x) = OR = -OR' = -\sin x \qquad \cos\theta = \cos(180 + x) = OQ = -OQ' = -\cos x$$

Hence $\quad \tan(180 + x) = \dfrac{\sin(180° + x)}{\cos(180° + x)} = \dfrac{\sin x}{\cos x} = +\tan x$ for all x.

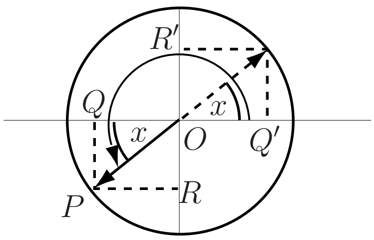
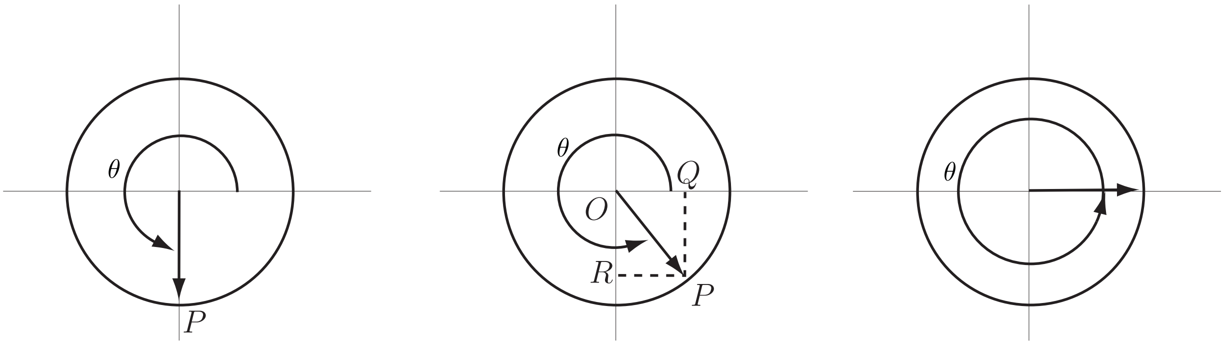


Figure 27: $\theta = 180° + x$

Key Point 9

$$\sin(180 + x) \equiv -\sin x \qquad \cos(180 + x) \equiv -\cos x \qquad \tan(180 + x) \equiv +\tan x$$

Fourth quadrant $(270° \leq \theta \leq 360°)$



$\theta = 270°$
$\cos\theta = \ 0$
$\sin\theta = -1$

$270° < \theta < 360°$
(alternatively $-90° < \theta < 0°$)
$\cos\theta = OQ > 0$
$\sin\theta = OR < 0$

$360°$
(results as for $0°$)

Figure 28

From Figure 28 the results in Key Point 10 should be clear.

Key Point 10

$$\cos(-x) \equiv \cos x \qquad \sin(-x) \equiv -\sin x \qquad \tan(-x) \equiv -\tan x.$$

Task

Write down (without using a calculator) the values of
$\sin 300°$, $\sin(-60°)$, $\cos 330°$, $\cos(-30°)$.

Describe the behaviour of $\cos\theta$ and $\sin\theta$ as θ increases from $270°$ to $360°$.

Your solution

Answer

$$\sin 300° = -\sin 60° = -\sqrt{3}/2 \qquad \cos 330° = \cos 30° = \sqrt{3}/2$$

$$\sin(-60°) = -\sin 60° = -\sqrt{3}/2 \qquad \cos(-30°) = \cos 30° = \sqrt{3}/2$$

$\cos\theta$ increases from 0 to 1 and $\sin\theta$ increases from -1 to 0 as θ increases from $270°$ to $360°$.

Rotation beyond the fourth quadrant $(360° < \theta)$

If the vector OP continues to rotate around the circle of unit radius then in the next complete rotation θ increases from $360°$ to $720°$. However, a θ value of, say, $405°$ is indistinguishable from one of $45°$ (just one extra complete revolution is involved).

So $\quad \sin(405°) = \sin 45° = \dfrac{1}{\sqrt{2}} \quad$ and $\quad \cos(405°) = \cos 45° = \dfrac{1}{\sqrt{2}}$

In general $\quad \sin(360° + x°) = \sin x°, \qquad \cos(360° + x°) = \cos x°$

Key Point 11

If n is any integer $\qquad \sin(x° + 360n°) \equiv \sin x° \qquad \cos(x° + 360n°) \equiv \cos x°$

or, since $360° \equiv 2\pi$ radians, $\qquad \sin(x + 2n\pi) \equiv \sin x \qquad \cos(x + 2n\pi) = \cos x$

We say that the functions $\sin x$ and $\cos x$ are **periodic** with period (in radian measure) of 2π.

3. Graphs of trigonometric functions

Graphs of sin θ and cos θ

Since we have defined both $\sin\theta$ and $\cos\theta$ in terms of the projections of the radius vector OP of a circle of **unit radius** it follows immediately that

$$-1 \leq \sin\theta \leq +1 \qquad \text{and} \qquad -1 \leq \cos\theta \leq +1 \qquad \text{for any value of } \theta.$$

We have discussed the behaviour of $\sin\theta$ and $\cos\theta$ in each of the four quadrants in the previous subsection.

Using all the above results we can draw the graphs of these two trigonometric functions. See Figure 29. We have labelled the horizontal axis using radians and have shown two **periods** in each case.

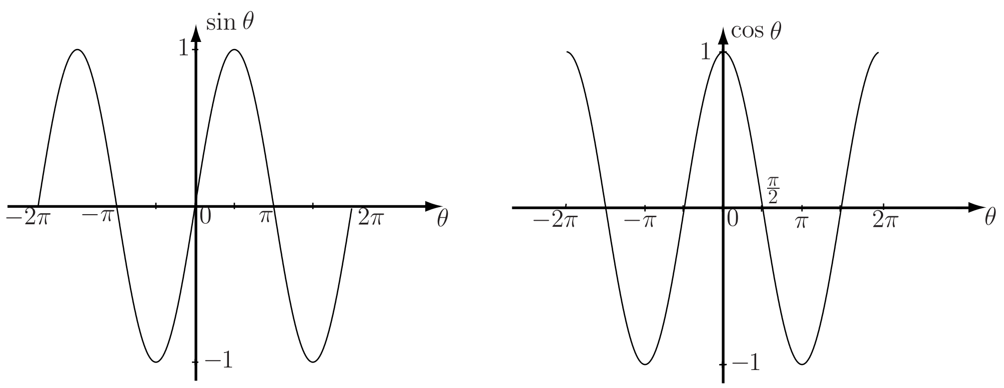


Figure 29

We have extended the graphs to negative values of θ using the relations $\sin(-\theta) = \sin\theta$, $\cos(-\theta) = \cos\theta$. Both graphs could be extended indefinitely to the left ($\theta \to -\infty$) and right ($\theta \to +\infty$).

Task

(a) Using the graphs in Figure 29 and the fact that $\tan\theta \equiv \sin\theta/\cos\theta$ calculate the values of $\tan 0,\ \tan\pi,\ \tan 2\pi$.

(b) For what values of θ is $\tan\theta$ undefined?

(c) State whether $\tan\theta$ is positive or negative in each of the four quadrants.

Your solution

(a)

(b)

(c)

Answer

(a)

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$\tan\pi = \frac{\sin\pi}{\cos\pi} = \frac{0}{-1} = 0$$

$$\tan 2\pi = \frac{\sin 2\pi}{\cos 2\pi} = \frac{0}{1} = 0$$

(b)

$\tan\theta$ is not be defined when $\cos\theta = 0$ i.e. when $\theta = \pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

(c)

1st quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$

2nd quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{+\text{ve}}{-\text{ve}} = -\text{ve}$

3rd quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\text{ve}}{-\text{ve}} = +\text{ve}$

4th quadrant: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$

The graph of $\tan\theta$

The graph of $\tan\theta$ against θ, for $-2\pi \leq \theta \leq 2\pi$ is then as in Figure 30. Note that whereas $\sin\theta$ and $\cos\theta$ have period 2π, $\tan\theta$ has period π.

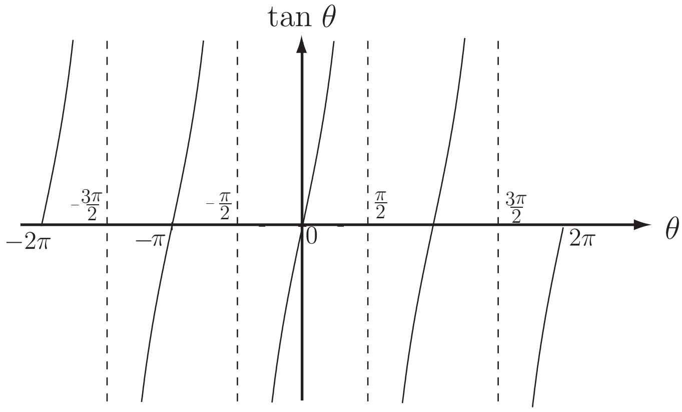


Figure 30

Task

On the following diagram showing the four quadrants mark which trigonometric quantities $\cos$, $\sin$, $\tan$, are positive in the four quadrants. One entry has been made already.

Your solution

	$\cos$

Answer

$\sin$	all
$\tan$	$\cos$

Engineering Example 4

Optical interference fringes due to a glass plate

Monochromatic light of intensity I_0 propagates in air before impinging on a glass plate (see Figure 31). If a screen is placed beyond the plate then a pattern is observed including alternate light and dark regions. These are **interference** fringes.

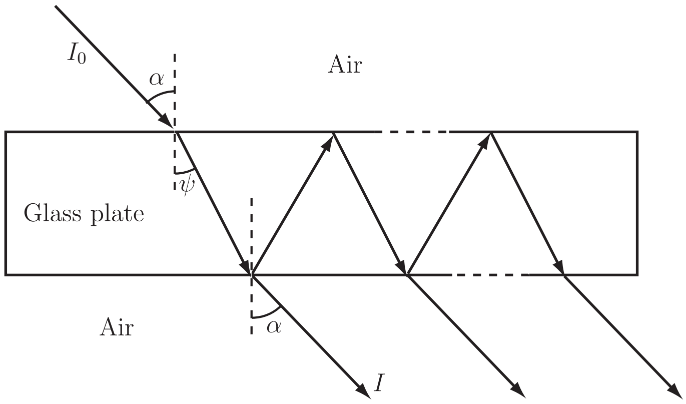


Figure 31: Geometry of a light ray transmitted and reflected through a glass plate

The intensity I of the light wave transmitted through the plate is given by

$$I = \frac{I_0|t|^4}{1 + |r|^4 - 2|r|^2\cos\theta}$$

where t and r are the complex transmission and reflection coefficients. The phase angle θ is the sum of

(i) a phase proportional to the incidence angle α and

(ii) a fixed phase lag due to multiple reflections.

The problem is to establish the form of the intensity pattern (i.e. the minima and maxima characteristics of interference fringes due to the plate), and deduce the shape and position θ of the fringes captured by a screen beyond the plate.

Solution

The intensity of the optical wave outgoing from the glass plate is given by

$$I = \frac{I_0|t|^4}{1 + |r|^4 - 2|r|^2\cos\theta} \qquad (1)$$

The light intensity depends solely on the variable θ as shown in equation (1), and the objective is to find the values θ that will minimize and maximize I. The angle θ is introduced in equation (1) through the function $\cos\theta$ in the denominator. We consider first the maxima of I.

Solution (contd.)

Light intensity maxima

I is maximum when the denominator is minimum. This condition is obtained when the factor $2|r|\cos\theta$ is maximum due to the minus sign in the denominator. As stated in Section 4.2, the maxima of $2|r|\cos\theta$ occur when $\cos\theta = +1$. Values of $\cos\theta = +1$ correspond to $\theta = 2n\pi$ where $n = \ldots -2, -1, 0, 1, 2, \ldots$ (see Section 4.5) and θ is measured in radians. Setting $\cos\theta = +1$ in equation (1) gives the intensity maxima

$$I_{\max} = \frac{I_0|t|^4}{1 + |r|^4 - 2|r|^2}.$$

Since the denominator can be identified as the square of $(1 + |r|^2)$, the final result for maximum intensity can be written as

$$I_{\max} = \frac{I_0|t|^4}{(1 - |r|^2)^2}. \qquad (2)$$

Light intensity minima

I is minimum when the denominator in (1) is maximum. As a result of the minus sign in the denominator, this condition is obtained when the factor $2|r|\cos\theta$ is minimum. The minima of $2|r|\cos\theta$ occur when $\cos\theta = -1$. Values of $\cos\theta = -1$ correspond to $\theta = \pi(2n+1)$ where $n = \ldots -2, -1, 0, 1, 2, \ldots$ (see Section 4.5). Setting $\cos\theta = -1$ in equation (1) gives an expression for the intensity minima

$$I_{\min} = \frac{I_0|t|^4}{1 + |r|^4 + 2|r|^2}.$$

Since the denominator can be recognised as the square of $(1 + |r|^2)$, the final result for minimum intensity can be written as

$$I_{\min} = \frac{I_0|t|^4}{(1 + |r|^2)^2} \qquad (3)$$

Interpretation

The interference fringes for intensity maxima or minima occur at constant angle θ and therefore describe concentric rings of alternating light and shadow as sketched in the figure below. From the centre to the periphery of the concentric ring system, the fringes occur in the following order

(a) a fringe of maximum light at the centre (bright dot for $\theta = 0$),

(b) a circular fringe of minimum light at angle $\theta = \pi$,

(c) a circular fringe of maximum light at 2π etc.

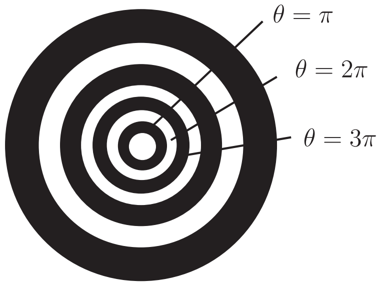


Figure 32: Sketch of interference fringes due to a glass plate

Exercises

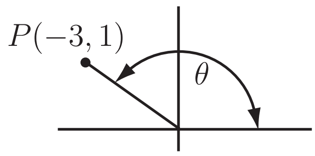
1. Express the following angles in radians (as multiples of π)

   (a) $120°$ (b) $20°$ (c) $135°$ (d) $300°$ (e) $-90°$ (f) $720°$

2. Express in degrees the following quantities which are in radians

   (a) $\dfrac{\pi}{2}$ (b) $\dfrac{3\pi}{2}$ (c) $\dfrac{5\pi}{6}$ (d) $\dfrac{11\pi}{9}$ (e) $-\dfrac{\pi}{8}$ (f) $\dfrac{1}{\pi}$

3. Obtain the **precise** values of all 6 trigonometric functions of the angle θ for the situation shown in the figure:



4. Obtain all the values of x between 0 and 2π such that

   (a) $\sin x = \dfrac{1}{\sqrt{2}}$ (b) $\cos x = \dfrac{1}{2}$ (c) $\sin x = -\dfrac{\sqrt{3}}{2}$ (d) $\cos x = -\dfrac{1}{\sqrt{2}}$ (e) $\tan x = 2$

   (f) $\tan x = -\dfrac{1}{2}$ (g) $\cos(2x + 60°) = 2$ (h) $\cos(2x + 60°) = \dfrac{1}{2}$

5. Obtain all the values of θ in the given domain satisfying the following quadratic equations

   (a) $2\sin^2\theta - \sin\theta = 0$ $\quad 0 \leq \theta \leq 360°$

   (b) $2\cos^2\theta + 7\cos\theta + 3 = 0$ $\quad 0 \leq \theta \leq 360°$

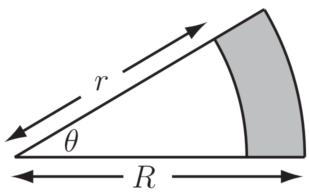
   (c) $4\sin^2\theta - 1 = 0$

6. (a) Show that the area A of a sector formed by a central angle θ radians in a circle of radius r is given by

   $$A = \frac{1}{2}r^2\theta.$$

   (Hint: By proportionality the ratio of the area of the sector to the total area of the circle equals the ratio of θ to the total angle at the centre of the circle.)

   (b) What is the value of the shaded area shown in the figure if θ is measured (i) in radians, (ii) in degrees?



7. Sketch, over $0 < \theta < 2\pi$, the graph of (a) $\sin 2\theta$ (b) $\sin\dfrac{1}{2}\theta$ (c) $\cos 2\theta$ (d) $\cos\dfrac{1}{2}\theta$.

   Mark the horizontal axis in radians in each case. Write down the period of $\sin 2\theta$ and the period of $\cos\dfrac{1}{2}\theta$.

Answers

1. (a) $\dfrac{2\pi}{3}$ (b) $\dfrac{\pi}{9}$ (c) $\dfrac{3\pi}{4}$ (d) $\dfrac{5\pi}{3}$ (e) $-\dfrac{\pi}{2}$ (f) 4π

2. (a) 15° (b) 270° (c) 150° (d) 220° (e) -22.5° (f) $\dfrac{180^\circ}{\pi^2}$

3. The distance of the point P from the origin is $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$. Then, since P lies on a circle radius $\sqrt{10}$ rather than a circle of unit radius:

$$\sin\theta = \frac{1}{\sqrt{10}} \qquad \operatorname{cosec}\theta = \sqrt{10}$$

$$\cos\theta = -\frac{3}{\sqrt{10}} \qquad \sec\theta = -\frac{\sqrt{10}}{3}$$

$$\tan\theta = \frac{1}{-3} = -\frac{1}{3} \qquad \cot\theta = -3$$

4. (a) $x = 45^\circ \left(\dfrac{\pi}{4}\text{ radians}\right)$ $x = 135^\circ \left(\dfrac{3\pi}{4}\right)$ (recall $\sin(180 - x) = \sin x$)

   (b) $x = 60^\circ \left(\dfrac{\pi}{3}\right)$ $x = 300^\circ \left(\dfrac{5\pi}{3}\right)$

   (c) $x = 240^\circ \left(\dfrac{4\pi}{3}\right)$ $x = 300^\circ \left(\dfrac{5\pi}{3}\right)$

   (d) $x = 135^\circ \left(\dfrac{3\pi}{4}\right)$ $x = 225^\circ \left(\dfrac{5\pi}{4}\right)$

   (e) $x = 63.43^\circ$ $x = 243.43^\circ$ (remember $\tan x$ has period 180° or π radians)

   (f) $x = 153.43^\circ$ $x = 333.43^\circ$

   (g) No solution !

   (h) $x = 0^\circ,\ 120^\circ,\ 180^\circ,\ 300^\circ,\ 360^\circ$

5. (a) $2\sin^2\theta - \sin\theta = 0$ so $\sin\theta(2\sin\theta - 1) = 0$ so $\sin\theta = 0$
   giving $\theta = 0^\circ,\ 180^\circ,\ 360^\circ$ or $\sin\theta = \dfrac{1}{2}$ giving $\theta = 30^\circ,\ 150^\circ$

   (b) $2\cos^2\theta + 7\cos\theta + 3 = 0$. With $x = \cos\theta$ we have $2x^2 + 7x + 3 = 0$ $(2x+1)(x+3) = 0$ (factorising) so $2x = -1$ or $x = -\dfrac{1}{2}$. The solution $x = -3$ is impossible since $x = \cos\theta$.

   The equation $x = \cos\theta = -\dfrac{1}{2}$ has solutions $\theta = 120^\circ,\ 240^\circ$

   (c) $4\sin^2\theta = 1$ so $\sin^2\theta = \dfrac{1}{4}$ i.e. $\sin\theta = \pm\,\dfrac{1}{2}$ giving $\theta = 30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ$

Answers continued

6. (a) Using the hint,

$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

from where we obtain $A = \dfrac{\pi r^2 \theta}{2\pi} = \dfrac{r^2\theta}{2}$

(b) With θ in radians the shaded area is

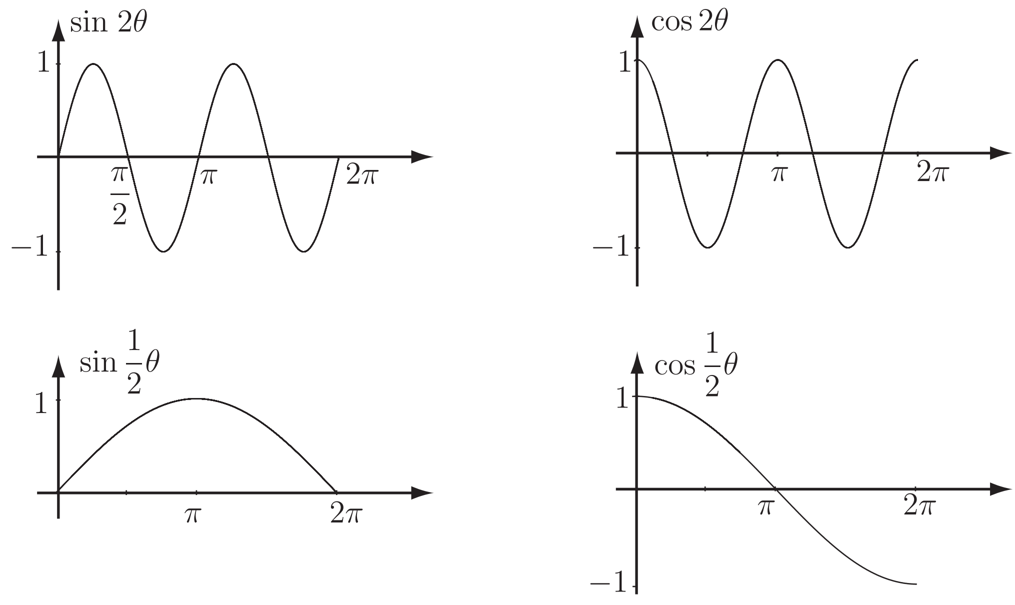
$$S = \frac{R^2\theta}{2} - \frac{r^2\theta}{2} = \frac{\theta}{2}(R^2 - r^2)$$

If θ is in degrees, then since x radians $= \dfrac{180x^\circ}{\pi}$ or $x^\circ = \dfrac{\pi x}{180}$ radians, we have

$$S = \frac{\pi\theta^\circ}{360^\circ}(R^2 - r^2)$$

7. The graphs of $\sin 2\theta$ and $\cos 2\theta$ are identical in form with those of $\sin\theta$ and $\cos\theta$ respectively but oscillate twice as rapidly.

The graphs of $\sin\dfrac{1}{2}\theta$ and $\cos\dfrac{1}{2}\theta$ oscillate half as rapidly as those of $\sin\theta$ and $\cos\theta$.



From the graphs $\sin 2\theta$ has period 2π and $\cos\frac{1}{2}\theta$ has period 4π. In general $\sin n\theta$ has period $2\pi/n$.

Trigonometric Identities

4.3

Introduction

A trigonometric identity is a relation between trigonometric expressions which is true for all values of the variables (usually angles). There are a very large number of such identities. In this Section we discuss only the most important and widely used. Any engineer using trigonometry in an application is likely to encounter some of these identities.

Prerequisites

Before starting this Section you should ...

- have a basic knowledge of the geometry of triangles

Learning Outcomes

On completion you should be able to ...

- use the main trigonometric identities
- use trigonometric identities to combine trigonometric functions

1. Trigonometric identities

An identity is a relation which is always true. To emphasise this the symbol '$\equiv$' is often used rather than '$=$'. For example, $(x+1)^2 \equiv x^2+2x+1$ (always true) but $(x+1)^2 = 0$ (only true for $x=-1$).

Task

(a) Using the exact values, evaluate $\sin^2\theta + \cos^2\theta$ for (i) $\theta = 30°$ (ii) $\theta = 45°$

[Note that $\sin^2\theta$ means $(\sin\theta)^2$, $\cos^2\theta$ means $(\cos\theta)^2$]

(b) Choose a non-integer value for θ and use a calculator to evaluate $\sin^2\theta+\cos^2\theta$.

Your solution

Answer

(a) (i) $\sin^2 30° + \cos^2 30° = \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{3}{4} = 1$

(ii) $\sin^2 45° + \cos^2 45° = \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2} + \dfrac{1}{2} = 1$

(b) The answer should be 1 whatever value you choose.

Key Point 12

For any value of θ

$$\sin^2\theta + \cos^2\theta \equiv 1 \qquad (5)$$

One way of proving the result in Key Point 12 is to use the definitions of $\sin\theta$ and $\cos\theta$ obtained from the circle of unit radius. Refer back to Figure 22 on page 23.
Recall that $\cos\theta = OQ$, $\sin\theta = OR = QP$. By Pythagoras' theorem

$$(OQ)^2 + (QP)^2 = (OP)^2 = 1$$

hence $\cos^2\theta + \sin^2\theta = 1$.
We have demonstrated the result (5) using an angle θ in the first quadrant but the result is true for any θ i.e. it is indeed an identity.

Task

By dividing the identity $\sin^2\theta + \cos^2\theta \equiv 1$ by (a) $\sin^2\theta$ (b) $\cos^2\theta$ obtain two further identities.

[Hint: Recall the definitions of $\text{cosec}\,\theta,\ \sec\theta,\ \cot\theta$.]

Your solution

Answer

(a) $\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$ (b) $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$

$\boxed{1 + \cot^2\theta \equiv \text{cosec}^2\theta}$ $\boxed{\tan^2\theta + 1 \equiv \sec^2\theta}$

Key Point 13 introduces two further important identities.

Key Point 13

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B \qquad (6)$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B \qquad (7)$$

Note carefully the addition sign in (6) but the subtraction sign in (7).

Further identities can readily be obtained from (6) and (7).
Dividing (6) by (7) we obtain

$$\tan(A+B) \equiv \frac{\sin(A+B)}{\cos(A+B)} \equiv \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

Dividing every term by $\cos A\ \cos B$ we obtain

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Replacing B by $-B$ in (6) and (7) and remembering that $\cos(-B) \equiv \cos B,\ \sin(-B) \equiv -\sin B$ we find

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

Task

Using the identities $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$ and $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$ obtain an expansion for $\tan(A - B)$:

Your solution

Answer

$$\tan(A - B) \equiv \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}.$$

Dividing every term by $\cos A \ \cos B$ gives

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The following identities are derived from those in Key Point 13.

Key Point 14

$$\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad (8)$$

$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B \qquad (9)$$

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B \qquad (10)$$

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B} \qquad (11)$$

Engineering Example 5

Amplitude modulation

Introduction

Amplitude Modulation (the AM in AM radio) is a method of sending electromagnetic signals of a certain frequency (signal frequency) at another frequency (carrier frequency) which may be better for transmission. Modulation can be represented by the multiplication of the carrier and modulating signals. To **demodulate** the signal the carrier frequency must be removed from the modulated signal.

Problem in words

(a) A single frequency of 200 Hz (message signal) is amplitude modulated with a carrier frequency of 2 MHz. Show that the modulated signal can be represented by the sum of two frequencies at $2 \times 10^6 \pm 200$ Hz

(b) Show that the modulated signal can be demodulated by using a locally generated carrier and applying a low-pass filter.

Mathematical statement of problem

(a) Express the message signal as $m = a\cos(\omega_m t)$ and the carrier as $c = b\cos(u_c t)$.

Assume that the modulation gives the product $mc = ab\cos(u_c t)\cos(\omega_m t)$.

Use trigonometric identities to show that

$$mc = ab\cos(\omega_c t)\cos(u_m t) = k_1\cos((\omega_c - u_m)t) + k_2\cos((\omega_c + u_m)t)$$

where k_1 and k_2 are constants.

Then substitute $\omega_c = 2 \times 10^6 \times 2\pi$ and $\omega_m = 200 \times 2\pi$ to calculate the two resulting frequencies.

(b) Use trigonometric identities to show that multiplying the modulated signal by $b\cos(u_c t)$ results in the lowest frequency component of the output having a frequency equal to the original message signal.

Mathematical analysis

(a) The message signal has a frequency of $f_m = 200$ Hz so $\omega_m = 2\pi f_c = 2\pi \times 200 = 400\pi$ radians per second.

The carrier signal has a frequency of $f_c = 2 \times 10^6$ Hz.

Hence $\omega_c = 2\pi f_c = 2\pi \times 2 \times 10^6 = 4 \times 10^6\pi$ radians per second.

So $mc = ab\cos(4 \times 10^6\pi t)\cos(400\pi t)$.

Key Point 13 includes the identity:

$$\cos(A + B) + \cos(A - B) \equiv 2\cos(A)\cos(B)$$

Rearranging gives the identity:

$$\cos(A)\cos(B) \equiv \tfrac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad (1)$$

Using (1) with $A = 4 \times 10^6 \pi t$ and $B = 400\pi t$ gives

$$\begin{aligned} mc &= ab(\cos(4 \times 10^6 \pi t)\cos(400\pi t) \\ &= ab(\cos(4 \times 10^6 \pi t + 400\pi t) + \cos(4 \times 10^6 \pi t - 400\pi t)) \\ &= ab(\cos(4000400\pi t) + \cos(3999600\pi t)) \end{aligned}$$

So the modulated signal is the **sum of two waves** with angular frequency of 4000400π and 3999600π radians per second corresponding to frequencies of $4000400\pi/(2\pi)$ and $39996000\pi/(2\pi)$, that is 2000200 Hz and 1999800 Hz i.e. $2 \times 10^6 \pm 200$ Hz.

(b) Taking identity (1) and multiplying through by $\cos(A)$ gives

$$\cos(A)\cos(A)\cos(B) \equiv \tfrac{1}{2}\cos(A)(\cos(A+B) + \cos(A-B))$$

so

$$\cos(A)\cos(A)\cos(B) \equiv \tfrac{1}{2}(\cos(A)\cos(A+B) + \cos(A)\cos(A-B)) \qquad (2)$$

Identity (1) can be applied to both expressions in the right-hand side of (2). In the first expression, using $A + B$ instead of 'B', gives

$$\cos(A)\cos(A+B) \equiv \frac{1}{2}(\cos(A+A+B) + \cos(A-A-B)) \equiv \frac{1}{2}(\cos(2A+B) + \cos(B))$$

where we have used $\cos(-B) \equiv \cos(B)$.

Similarly, in the second expression, using $A - B$ instead of 'B', gives

$$\cos(A)\cos(A-B) \equiv \tfrac{1}{2}(\cos(2A-B) + \cos(B))$$

Together these give:

$$\begin{aligned} \cos(A)\cos(A)\cos(B) &\equiv \frac{1}{2}(\cos(2A+B) + \cos(B) + \cos(2A-B) + \cos(B)) \\ &\equiv \cos(B) + \frac{1}{2}(\cos(2A+B) + \cos(2A-B)) \end{aligned}$$

With $A = 4 \times 10^6 \pi t$ and $B = 400\pi t$ and substituting for the given frequencies, the modulated signal multiplied by the original carrier signal gives

$$ab^2 \cos(4 \times 10^6 \pi t)\cos(4 \times 10^6 \pi t)\cos(400\pi t) =$$
$$ab^2 \cos(2\pi \times 200t) + \tfrac{1}{2}ab^2(\cos(2 \times 4 \times 10^6 \pi t + 400\pi t) + \cos(2 \times 4 \times 10^6 \pi t - 400\pi t))$$

The last two terms have frequencies of $4 \times 10^6 \pm 200$ Hz which are sufficiently high that a low-pass filter would remove them and leave only the term

$$ab^2 \cos(2\pi \times 200t)$$

which is the original message signal multiplied by a constant term.

Interpretation

Amplitude modulation of a single frequency message signal (f_m) with a single frequency carrier signal (f_c) can be shown to be equal to the sum of two cosines with frequencies $f_c \pm f_m$. Multiplying the modulated signal by a locally generated carrier signal and applying a low-pass filter can reproduce the frequency, f_m, of the message signal.

This is known as **double side band amplitude modulation**.

Example 2

Obtain expressions for $\cos\theta$ in terms of the sine function and for $\sin\theta$ in terms of the cosine function.

Solution

Using (9) with $A = \theta,\ B = \dfrac{\pi}{2}$ we obtain

$$\cos\left(\theta - \frac{\pi}{2}\right) \equiv \cos\theta\ \cos\left(\frac{\pi}{2}\right) + \sin\theta\ \sin\left(\frac{\pi}{2}\right) \equiv \cos\theta\ (0) + \sin\theta\ (1)$$

i.e. $\sin\theta \equiv \cos\left(\theta - \dfrac{\pi}{2}\right) \equiv \cos\left(\dfrac{\pi}{2} - \theta\right)$

This result explains why the graph of $\sin\theta$ has exactly the same shape as the graph of $\cos\theta$ but it is shifted to the right by $\dfrac{\pi}{2}$. (See Figure 29 on page 28). A similar calculation using (6) yields the result

$$\cos\theta \equiv \sin\left(\theta + \frac{\pi}{2}\right).$$

Double angle formulae

If we put $B = A$ in the identity given in (6) we obtain Key Point 15:

Key Point 15

$\sin 2A \equiv \sin A\cos A + \cos A\sin A$ so $\sin 2A \equiv 2\sin A\cos A$ (12)

Task

Substitute $B = A$ in identity (7) in Key Point 13 on page 38 to obtain an identity for $\cos 2A$. Using $\sin^2 A + \cos^2 A \equiv 1$ obtain two alternative forms of the identity.

Your solution

Answer

Using (7) with $B \equiv A$

$$\begin{aligned} \cos(2A) &\equiv (\cos A)(\cos A) - (\sin A)(\sin A) \\ \therefore \quad \cos(2A) &\equiv \cos^2 A - \sin^2 A \qquad (13) \end{aligned}$$

Substituting for $\sin^2 A$ in (13) we obtain

$$\begin{aligned} \cos 2A &\equiv \cos^2 A - (1 - \cos^2 A) \\ &\equiv 2\cos^2 A - 1 \qquad (14) \end{aligned}$$

Alternatively substituting for $\cos^2 A$ in (13)

$$\begin{aligned} \cos 2A &\equiv (1 - \sin^2 A) - \sin^2 A \\ \cos 2A &\equiv 1 - 2\sin^2 A \qquad (15) \end{aligned}$$

Task

Use (14) and (15) to obtain, respectively, $\cos^2 A$ and $\sin^2 A$ in terms of $\cos 2A$.

Your solution

Answer

From (14) $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$. From (15) $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$.

Use (12) and (13) to obtain an identity for $\tan 2A$ in terms of $\tan A$.

Your solution

Answer

$$\tan 2A \equiv \frac{\sin 2A}{\cos 2A} \equiv \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Dividing numerator and denominator by $\cos^2 A$ we obtain

$$\tan 2A \equiv \frac{2\dfrac{\sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}} \equiv \frac{2\tan A}{1 - \tan^2 A} \qquad (16)$$

Half-angle formulae

If we replace A by $\dfrac{A}{2}$ and, consequently $2A$ by A, in (12) we obtain

$$\sin A \equiv 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \qquad (17)$$

Similarly from (13)

$$\cos A \equiv 2\cos^2\left(\frac{A}{2}\right) - 1. \qquad (18)$$

These are examples of **half-angle formulae**. We can obtain a half-angle formula for $\tan A$ using (16). Replacing A by $\dfrac{A}{2}$ and $2A$ by A in (16) we obtain

$$\tan A \equiv \frac{2\tan\left(\dfrac{A}{2}\right)}{1 - \tan^2\left(\dfrac{A}{2}\right)} \qquad (19)$$

Other formulae, useful for integration when trigonometric functions are present, can be obtained using (17), (18) and (19) shown in the Key Point 16.

Key Point 16

If $t = \tan\left(\frac{A}{2}\right)$ then

$$\sin A = \frac{2t}{1+t^2} \qquad (20)$$

$$\cos A = \frac{1-t^2}{1+t^2} \qquad (21)$$

$$\tan A = \frac{2t}{1-t^2} \qquad (22)$$

Sum of two sines and sum of two cosines

Finally, in this Section, we obtain results that are widely used in areas of science and engineering such as vibration theory, wave theory and electric circuit theory.
We return to the identities (6) and (9)

$$\begin{aligned} \sin(A+B) &\equiv \sin A\cos B + \cos A \sin B \\ \sin(A-B) &\equiv \sin A \cos B - \cos A \sin B \end{aligned}$$

Adding these identities gives

$$\sin(A+B) + \sin(A-B) \equiv 2\sin A\cos B \qquad (23)$$

Subtracting the identities produces

$$\sin(A+B) - \sin(A-B) \equiv 2\cos A \sin B \qquad (24)$$

It is now convenient to let $C = A+B$ and $D = A-B$ so that

$$A = \frac{C+D}{2} \quad \text{and} \quad B = \frac{C-D}{2}$$

Hence (23) becomes

$$\sin C + \sin D \equiv 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad (25)$$

Similarly (24) becomes

$$\sin C - \sin D \equiv 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \qquad (26)$$

Use (7) and (10) to obtain results for the sum and difference of two cosines.

Your solution

Answer

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B \quad \text{and} \quad \cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\therefore \quad \cos(A+B) + \cos(A-B) \equiv 2\cos A\cos B$$

$$\cos(A+B) - \cos(A-B) \equiv -2\sin A\sin B$$

Hence with $C = A + B$ and $D = A - B$

$$\cos C + \cos D \equiv 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad (27)$$

$$\cos C - \cos D \equiv -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \qquad (28)$$

Summary

In this Section we have covered a large number of trigonometric identities. The most important of them and probably the ones most worth memorising are given in the following Key Point.

Key Point 17

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &\equiv 1 \\
\sin 2\theta &\equiv 2\sin\theta\cos\theta \\
\cos 2\theta &\equiv \cos^2\theta - \sin^2\theta \\
&\equiv 2\cos^2\theta - 1 \\
&\equiv 1 - 2\sin^2\theta \\
\sin(A \pm B) &\equiv \sin A\cos B \pm \cos A\sin B \\
\cos(A \pm B) &\equiv \cos A\cos B \mp \sin A\sin B
\end{aligned}$$

Task

A projectile is fired from the ground with an initial speed u m s^{-1} at an angle of elevation $\alpha°$. If air resistance is neglected, the vertical height, y m, is related to the horizontal distance, x m, by the equation

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2} \qquad \text{where } g \text{ m s}^{-2} \text{ is the gravitational constant.}$$

[This equation is derived in HELM 34 Modelling Motion pages 16-17.]

(a) Confirm that $y = 0$ when $x = 0$:

Your solution

Answer

When $y = 0$, the left-hand side of the equation is zero. Since x appears in both of the terms on the right-hand side, when $x = 0$, the right-hand side is zero.

(b) Find an expression for the value of x other than $x = 0$ at which $y = 0$ and state how this value is related to the maximum range of the projectile:

Your solution

Answer

When $y = 0$, the equation can be written $\quad \dfrac{gx^2\sec^2\alpha}{2u^2} - x\tan\alpha = 0$

If $x = 0$ is excluded from consideration, we can divide through by x and rearrange to give

$$\frac{gx\sec^2\alpha}{2u^2} = \tan\alpha$$

To make x the subject of the equation we need to multiply both sides by $\dfrac{2u^2}{g\sec^2\alpha}$.

Given that $1/\sec^2\alpha \equiv \cos^2\alpha$, $\quad \tan\alpha \equiv \sin\alpha/\cos\alpha$ and $\sin 2\alpha \equiv 2\sin\alpha\cos\alpha$, this results in

$$x = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

This represents the maximum range.

(c) Find the value of x for which the value of y would be a maximum and thereby obtain an expression for the maximum height:

Your solution

Answer

If air resistance is neglected, we can assume that the parabolic path of the projectile is symmetrical about its highest point. So the highest point will occur at half the maximum range i.e. where

$$x = \frac{u^2 \sin 2\alpha}{2g}$$

Substituting this expression for x in the equation for y gives

$$y = \left(\frac{u^2 \sin 2\alpha}{2g}\right) \tan\alpha - \left(\frac{u^2 \sin 2\alpha}{2g}\right)^2 \frac{g\sec^2\alpha}{2u^2}$$

Using the same trigonometric identities as before,

$$y = \frac{u^2 \sin^2\alpha}{g} - \frac{u^2 \sin^2\alpha}{2g} = \frac{u^2 \sin^2\alpha}{2g}$$ This represents the maximum height.

(d) Assuming $u = 20$ m s^{-1}, $\alpha = 60^\circ$ and $g = 10$ m s^{-2}, find the maximum value of the range and the horizontal distances travelled when the height is 10 m:

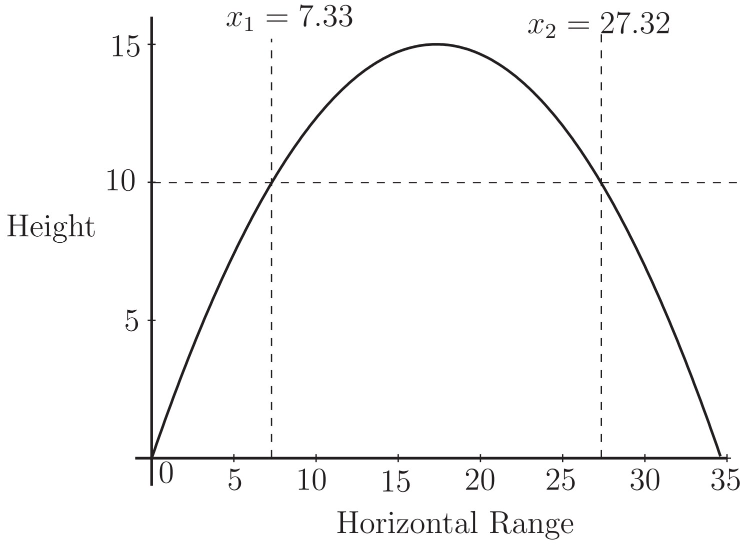
Your solution

Answer

Substitution of $u = 20,\ \alpha = 60,\ g = 10$ and $y = 10$ in the original equation gives a quadratic for x:

$$10 = 1.732x - 0.05x^2 \quad \text{or} \quad 0.05x^2 - 1.732x + 10 = 0$$

Solution of this quadratic yields $x = 7.33$ or $x = 27.32$ as the two horizontal ranges at which $y = 10$. These values are illustrated in the diagram below which shows the complete trajectory of the projectile.



Exercises

1. Show that $\sin t \sec t \equiv \tan t$.

2. Show that $(1 + \sin t)(1 + \sin(-t)) \equiv \cos^2 t$.

3. Show that $\dfrac{1}{\tan\theta + \cot\theta} \equiv \dfrac{1}{2}\sin 2\theta$.

4. Show that $\sin^2(A + B) - \sin^2(A - B \equiv \sin 2A \sin 2B$.

   (Hint: the left-hand side is the difference of two squared quantities.)

5. Show that $\dfrac{\sin 4\theta + \sin 2\theta}{\cos 4\theta + \cos 2\theta} \equiv \tan 3\theta$.

6. Show that $\cos^4 A - \sin^4 A \equiv \cos 2A$

7. Express each of the following as the sum (or difference) of 2 sines (or cosines)

   (a) $\sin 5x \cos 2x$ (b) $8\cos 6x \cos 4x$ (c) $\dfrac{1}{3}\sin\dfrac{1}{2}x\cos\dfrac{3}{2}x$

8. Express (a) $\sin 3\theta$ in terms of $\cos\theta$. (b) $\cos 3\theta$ in terms of $\cos\theta$.

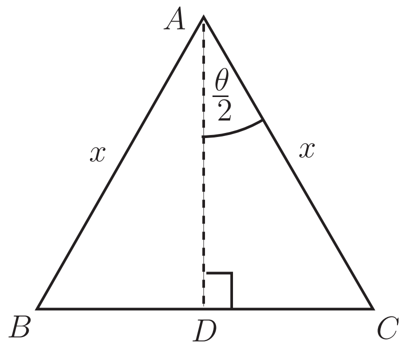
9. By writing $\cos 4x$ as $\cos 2(2x)$, or otherwise, express $\cos 4x$ in terms of $\cos x$.

10. Show that $\tan 2t \equiv \dfrac{2\tan t}{2 - \sec^2 t}$.

11. Show that $\dfrac{\cos 10t - \cos 12t}{\sin 10t + \sin 12t} \equiv \tan t$.

12. Show that the area of an isosceles triangle with equal sides of length x is $\dfrac{x^2}{2}\sin\theta$

    where θ is the angle between the two equal sides. Hint: use the following diagram:



Answers

1. $\sin t.\text{sec}t \equiv \sin t.\dfrac{1}{\cos t} \equiv \dfrac{\sin t}{\cos t} \equiv \tan t.$

2. $(1+\sin t)(1+\sin(-t)) \equiv (1+\sin t)(1-\sin t) \equiv 1-\sin^2 t \equiv \cos^2 t$

3. $\dfrac{1}{\tan\theta+\cos\theta} \equiv \dfrac{1}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}} \equiv \dfrac{1}{\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}} \equiv \dfrac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta} \equiv \sin\theta\cos\theta \equiv \dfrac{1}{2}\sin 2\theta$

4. Using the hint and the identity $x^2-y^2 \equiv (x-y)(x+y)$ we have

$$\sin^2(A+B)-\sin^2(A-B) \equiv (\sin(A+B)-\sin(A-B))(\sin(A+B)+\sin(A-B))$$

The first bracket gives

$$\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B) \equiv 2\cos A\sin B$$

Similarly the second bracket gives $2\sin A\cos B.$

Multiplying we obtain $(2\cos A\sin A)(2\cos B\sin B) \equiv \sin 2A.\sin 2B$

5. $\dfrac{\sin 4\theta+\sin 2\theta}{\cos 4\theta+\cos 2\theta} \equiv \dfrac{2\sin 3\theta\cos\theta}{2\cos 3\theta\cos\theta} \equiv \dfrac{\sin 3\theta}{\cos 3\theta} \equiv \tan 3\theta$

6.

$$\begin{aligned}\cos^4 A-\sin^4 A \equiv (\cos A)^4-(\sin A)^4 &\equiv (\cos^2 A)^2-(\sin^2 A)^2\\ &\equiv (\cos^2 A-\sin^2 A)(\cos^2 A+\sin^2 A)\\ &\equiv \cos^2 A-\sin^2 A \equiv \cos 2A\end{aligned}$$

7. (a) Using $\sin A+\sin B \equiv 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

Clearly here $\dfrac{A+B}{2}=5x \qquad \dfrac{A-B}{2}=2x$ giving $A=7x \quad B=3x$

$\therefore \qquad \sin 5x\cos 2x \equiv \dfrac{1}{2}(\sin 7x+\sin 3x)$

(b) Using $\cos A+\cos B \equiv 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right).$

With $\dfrac{A+B}{2}=6x \qquad \dfrac{A-B}{2}=4x$ giving $A=10x \quad B=2x$

$\therefore \quad 8\cos 6x\cos 4x \equiv 4(\cos 6x+\cos 2x)$

(c) $\dfrac{1}{3}\sin\left(\dfrac{1}{2}x\right)\cos\left(\dfrac{3x}{2}\right) \equiv \dfrac{1}{6}(\sin 2x-\sin x)$

Answers

8.

(a)
$$\begin{aligned}\sin 3\theta &\equiv \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &\equiv 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta\\ &\equiv 3\sin\theta\cos^2\theta - \sin^3\theta\\ &\equiv 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \equiv 3\sin\theta - 4\sin^3\theta\end{aligned}$$

(b)
$$\begin{aligned}\cos 3\theta &\equiv \cos(2\theta+\theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta\\ &\equiv \cos^3\theta - 3\sin^2\theta\cos\theta\\ &\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta\\ &\equiv 4\cos^3\theta - 3\cos\theta\end{aligned}$$

9.

$$\begin{aligned}\cos 4x = \cos 2(2x) &\equiv 2\cos^2(2x) - 1\\ &\equiv 2(\cos 2x)^2 - 1\\ &\equiv 2(2\cos^2 x - 1)^2 - 1\\ &\equiv 2(4\cos^4 x - 4\cos^2 x + 1) - 1 \equiv 8\cos^4 x - 8\cos^2 x + 1.\end{aligned}$$

10. $\tan 2t \equiv \dfrac{2\tan t}{1-\tan^2 t} \equiv \dfrac{2\tan t}{1-(\sec^2 t - 1)} \equiv \dfrac{2\tan t}{2-\sec^2 t}$

11. $\cos 10t - \cos 12t \equiv 2\sin 11t\sin t \qquad \sin 10t + \sin 12t \equiv 2\sin 11t\cos(-t)$

$$\therefore \quad \frac{\cos 10t - \cos 12t}{\sin 10t + \sin 12t} \equiv \frac{\sin t}{\cos(-t)} \equiv \frac{\sin t}{\cos t} \equiv \tan t$$

12. The right-angled triangle ACD has area $\dfrac{1}{2}(CD)(AD)$

$$\text{But} \quad \sin\left(\frac{\theta}{2}\right) = \frac{CD}{x} \qquad \therefore \quad CD = x\sin\left(\frac{\theta}{2}\right)$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{AD}{x} \qquad \therefore \quad AD = x\cos\left(\frac{\theta}{2}\right)$$

$$\therefore \quad \text{area of } \Delta ACD = \frac{1}{2}x^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{4}x^2\sin\theta$$

$$\therefore \quad \text{area of } \Delta ABC = 2\times\text{area of } \Delta ACD = \frac{1}{2}x^2\sin\theta$$

Applications of Trigonometry to Triangles

4.4

Introduction

We originally introduced trigonometry using right-angled triangles. However, the subject has applications in dealing with **any** triangles such as those that might arise in surveying, navigation or the study of mechanisms.

In this Section we show how, given certain information about a triangle, we can use appropriate rules, called the **Sine rule** and the **Cosine rule**, to fully 'solve the triangle' i.e. obtain the lengths of all the sides and the size of all the angles of that triangle.

Prerequisites

Before starting this Section you should . . .

- have a knowledge of the basics of trigonometry
- be aware of the standard trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- use trigonometry in everyday situations
- fully determine all the sides and angles and the area of any triangle from partial information

1. Applications of trigonometry to triangles

Area of a triangle

The area S of any triangle is given by $S = \frac{1}{2} \times (\text{base}) \times (\text{perpendicular height})$ where 'perpendicular height' means the perpendicular distance from the side called the 'base' to the opposite vertex. Thus for the right-angled triangle shown in Figure 33(a) $\quad S = \frac{1}{2}\ b\ a$. For the obtuse-angled triangle shown in Figure 33(b) the area is $S = \frac{1}{2}bh$.

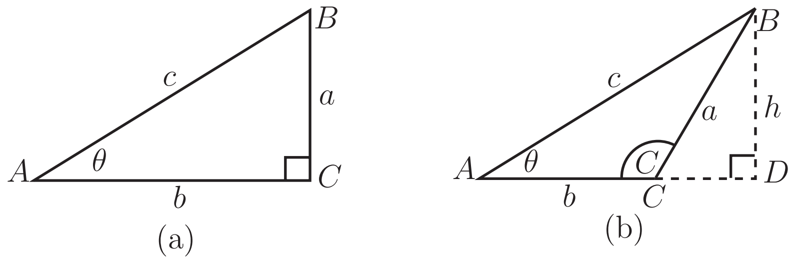


Figure 33

If we use C to denote the angle ACB in Figure 33(b) then

$$\sin(180 - C) = \frac{h}{a} \qquad \text{(triangle } BCD \text{ is right-angled)}$$

$$\therefore \quad h = a\sin(180 - C) = a\sin C \quad \text{(see the graph of the sine wave or expand } \sin(180 - c))$$

$$\therefore \quad \boxed{S = \frac{1}{2}\ b\ a\sin C} \qquad 1(a)$$

By other similar constructions we could demonstrate that

$$\boxed{S = \frac{1}{2}\ a\ c\ \sin B} \qquad 1(b)$$

and

$$\boxed{S = \frac{1}{2}\ b\ c\ \sin A} \qquad 1(c)$$

Note the pattern here: in each formula for the area the angle involved is the one **between** the sides whose lengths occur in that expression.
Clearly if C is a right-angle (so $\sin C = 1$) then

$$S = \frac{1}{2}\ b\ a \quad \text{as for Figure 33(a).}$$

Note: from now on we will not generally write '$\equiv$' but use the more usual '$=$'.

The Sine rule

The Sine rule is a formula which, if we are given certain information about a triangle, enables us to fully 'solve the triangle' i.e. obtain the lengths of all three sides and the value of all three angles. To show the rule we note that from the formulae (1a) and (1b) for the area S of the triangle ABC in Figure 33 we have

$$ba\sin C = ac\sin B \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly using (1b) and (1c)

$$ac\sin B = bc\sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Key Point 18

The Sine Rule

For **any** triangle ABC where a is the length of the side opposite angle A, b the side length opposite angle B and c the side length opposite angle C states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use of the Sine rule

To be able to fully determine all the angles and sides of a triangle it follows from the Sine rule that we must know

either two angles and one side : (knowing two angles of a triangle really means that all three are known since the sum of the angles is $180°$)

or two sides and an angle **opposite** one of those two sides.

Example 3

Solve the triangle ABC given that $a = 32$ cm, $b = 46$ cm and angle $B = 63.25°$.

Solution

Using the first pair of equations in the Sine rule (Key Point 18) we have

$$\frac{32}{\sin A} = \frac{46}{\sin 63.25°} \qquad \therefore \qquad \sin A = \frac{32}{46}\sin 63.25° = 0.6212$$

so $\quad A = \sin^{-1}(0.6212) = 38.4° \quad$ (by calculator)

Solution (contd.)

You should, however, note carefully that because of the form of the graph of the sine function there are **two** angles between 0° and 180° which have the same value for their sine i.e. x and $(180 - x)$. See Figure 34.

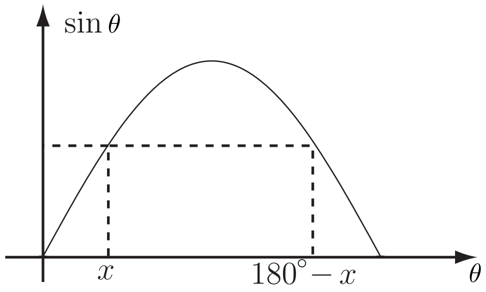


Figure 34

In our example

$$A = \sin^{-1}(0.6212) = 38.4^\circ$$

or

$$A = 180^\circ - 38.4^\circ = 141.6^\circ.$$

However since we are given that angle B is 63.25°, the value of 141.6° for angle A is clearly impossible.

To complete the problem we simply note that

$$C = 180^\circ - (38.4^\circ + 63.25^\circ) = 78.35^\circ$$

The remaining side c is calculated from the Sine rule, using either a and $\sin A$ or b and $\sin B$.

Task

Find the length of side c in Example 3.

Your solution

Answer

Using, for example, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

we have $c = a\dfrac{\sin C}{\sin A} = 32 \times \dfrac{\sin 78.35^\circ}{0.6212} = \dfrac{32 \times 0.9794}{0.6212} = 50.45$ cm.

The ambiguous case

When, as in Example 3, we are given two sides and the non-included angle of a triangle, particular care is required.

Suppose that sides b and c and the angle B are given. Then the angle C is given by the Sine rule as

$$\sin C = c\frac{\sin B}{b}$$

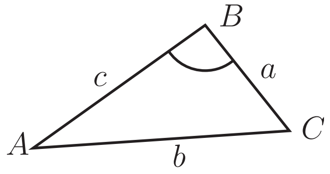


Figure 35

Various cases can arise:

(i) $c \sin B > b$

This implies that $\dfrac{c \sin B}{b} > 1$ in which case no triangle exists since $\sin C$ cannot exceed 1.

(ii) $c \sin B = b$

In this case $\sin C = \dfrac{c \sin B}{b} = 1$ so $C = 90°$.

(iii) $c \sin B < b$

Hence $\sin C = \dfrac{c \sin B}{b} < 1.$

As mentioned earlier there are two possible values of angle C in the range 0 to $180°$, one acute angle $(< 90°)$ and one obtuse (between $90°$ and $180°$.) These angles are $C_1 = x$ and $C_2 = 180 - x$. See Figure 36.

If the given angle B is greater than $90°$ then the obtuse angle C_2 is not a possible solution because, of course, a triangle cannot possess two obtuse angles.

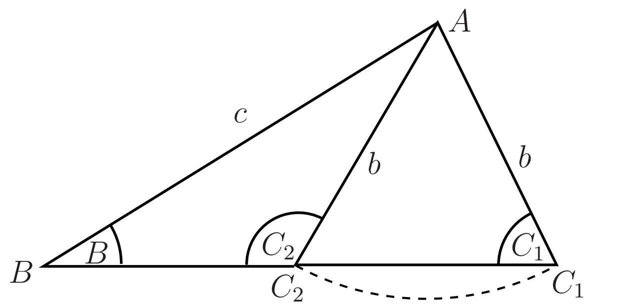


Figure 36

For B less than $90°$ there are still two possibilities.

If the given side b is greater than the given side c, the obtuse angle solution C_2 is not possible because then the larger angle would be opposite the smaller side. (This was the situation in Example 3.)

The final case

$$b < c, \qquad B < 90°$$

does give rise to two possible values C_1, C_2 of the angle C and is referred to as the **ambiguous case**. In this case there will be two possible values a_1 and a_2 for the third side of the triangle corresponding to the two angle values

$$\begin{aligned} A_1 &= 180° - (B + C_1) \\ A_2 &= 180° - (B + C_2) \end{aligned}$$

Task

Show that two triangles fit the following data for a triangle ABC:

$a = 4.5$ cm $\qquad b = 7$ cm $\qquad A = 35°$

Obtain the sides and angle of both possible triangles.

Your solution

Answer

We have, by the Sine rule, $\quad \sin B = \dfrac{b \sin A}{a} = \dfrac{7 \sin 35°}{4.5} = 0.8922$

So $B = \sin^{-1} 0.8922 - 63.15°$ (by calculator) or $180 - 63.15° = 116.85°$.

In this case, **both** values of B are indeed possible since both values are larger than angle A (side b is longer than side a). This is the ambiguous case with two possible triangles.

$B = B_1 = 63.15°$

$C = C_1 = 81.85°$

$c = c_1$ where $\dfrac{c_1}{\sin 81.85°} = \dfrac{4.5}{\sin 35°}$

$$c_1 = \frac{4.5 \times 0.9899}{0.5736}$$

$$= 7.766 \text{ cm}$$

$B = B_2 = 116.85°$

$C = C_2 = 28.15°$

$c = c_2$ where $\dfrac{c_2}{\sin 28.15} = \dfrac{4.5}{\sin 35°}$

$$c_2 = \frac{4.5 \times 0.4718}{0.5736}$$

$$= 3.701 \text{ cm}$$

You can clearly see that we have one acute angled triangle AB_1C_1 and one obtuse angled AB_2C_2 corresponding to the given data.

The Cosine rule

The Cosine rule is an alternative formula for 'solving a triangle' ABC. It is particularly useful for the case where the Sine rule cannot be used, i.e. when two sides of the triangle are known together with the angle **between** these two sides.

Consider the two triangles ABC shown in Figure 37.

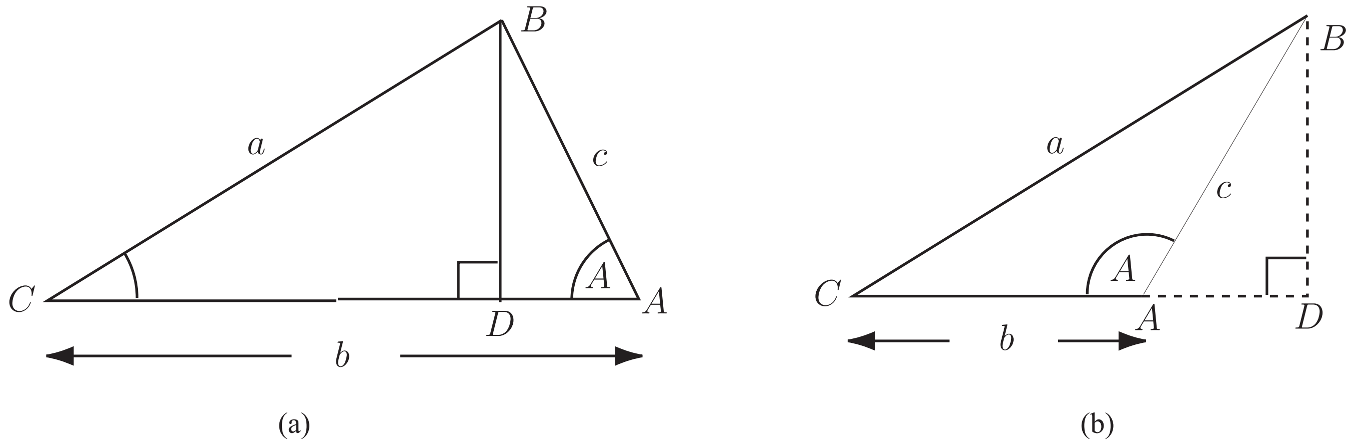


Figure 37

In Figure 37(a) using the right-angled triangle ABD, $BD = c\sin A$.

In Figure 37(b) using the right-angled triangle ABD, $BD = c\sin(\pi - A) = c\sin A$.

In Figure 37(a) $DA = c\cos A \quad \therefore \quad CD = b - c\cos A$

In Figure 37(b) $DA = c\cos(180 - A) = -c\cos A \quad \therefore \quad CD = b + AD = b - c\cos A$

In both cases, in the right-angled triangle BDC

$$(BC)^2 = (CD)^2 + (BD)^2$$

So, using the above results,

$$a^2 = (b - c\cos A)^2 + c^2(\sin A)^2 = b^2 - 2bc\cos A + c^2(\cos^2 A + \sin^2 A)$$

giving

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A} \qquad (3)$$

Equation (3) is one form of the Cosine rule. Clearly it can be used, as we stated above, to calculate the side a if the sides b and c and the **included** angle A are known.

Note that if $A = 90^\circ$, $\cos A = 0$ and (3) reduces to Pythagoras' theorem.

Two similar formulae to (3) for the Cosine rule can be similarly derived - see following Key Point:

Key Point 19

Cosine Rule

For any triangle with sides a, b, c and corresponding angles A, B, C

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = c^2 + a^2 - 2ca\cos B \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$c^2 = a^2 + b^2 - 2bc\cos C \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 4

Solve the triangle where $b = 7.00$ cm, $c = 3.59$ cm, $A = 47°$.

Solution

Since two sides and the angle A between these sides is given we must first use the Cosine rule in the form (3a):

$$\begin{aligned} a^2 &= (7.00)^2 + (3.59)^2 - 2(7.00)(3.59)\cos 47° = 49 + 12.888 - 34.277 = 27.610 \\ \text{so} \quad a &= \sqrt{27.610} = 5.255 \text{ cm.} \end{aligned}$$

We can now most easily use the Sine rule to solve one of the remaining angles:

$$\frac{7.00}{\sin B} = \frac{5.255}{\sin 47°} \quad \text{so} \quad \sin B = \frac{7.00 \sin 47°}{5.255} = 0.9742$$

from which $B = B_1 = 76.96°$ or $B = B_2 = 103.04°$.

At this stage it is not obvious which value is correct or whether this is the ambiguous case and both values of B are possible.

The two possible values for the remaining angle C are

$$C_1 = 180° - (47° + 76.96°) = 56.04°$$

$$C_2 = 180° - (47 + 103.04) = 29.96°$$

Since for the sides of this triangle $b > a > c$ then similarly for the angles we must have $B > A > C$ so the value $C_2 = 29.96°$ is the correct one for the third side.

The Cosine rule can also be applied to some triangles where the lengths a, b and c of the three sides are known and the only calculations needed are finding the angles.

Task

A triangle ABC has sides

$a = 7$cm $b = 11$ cm $c = 12$ cm.

Obtain the values of all the angles of the triangle. (Use Key Point 19.)

Your solution

Answer

Suppose we find angle A first using the following formula from Key Point 19

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Here $\cos A = \dfrac{11^2 + 12^2 - 7^2}{2 \times 11 \times 12} = 0.818$ so $A = \cos^{-1}(0.818) = 35.1^\circ$

(There is no other possibility between 0° and 180° for A. No 'ambiguous case' arises using the Cosine rule!)

Another angle B or C could now be obtained using the Sine rule or the Cosine rule.

Using the following formula from Key Point 19:

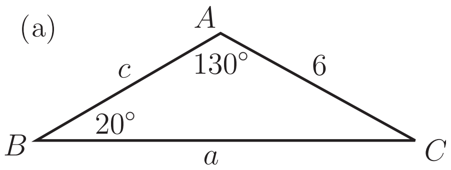
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{12^2 + 7^2 - 11^2}{2 \times 12 \times 7} = 0.429 \quad \text{so} \quad B = \cos^{-1}(0.429) = 64.6^\circ$$

Since $A + B + C = 180^\circ$ we can deduce $C = 80.3^\circ$

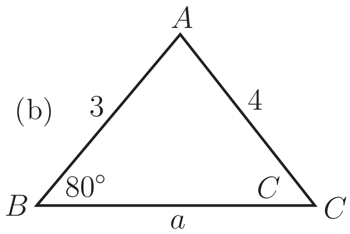
Exercises

1. Determine the remaining angles and sides for the following triangles:

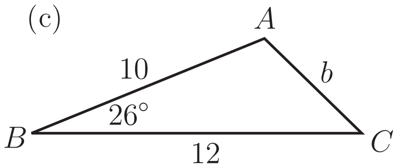
(a)



(b)



(c)

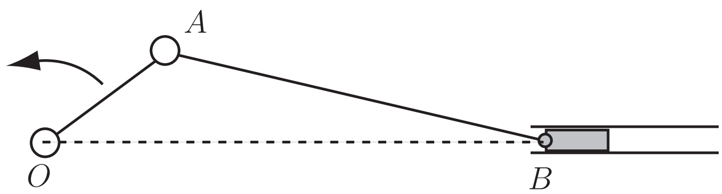


(d) The triangles ABC with $B = 50°,\ b = 5,\ c = 6$. (Take special care here!)

2. Determine all the angles of the triangles ABC where the sides have lengths $a = 7,\ b = 66$ and $c = 9$

3. Two ships leave a port at 8.00 am, one travelling at 12 knots (nautical miles per hour) the other at 10 knots. The faster ship maintains a bearing of $N47°W$, the slower one a bearing $S20°W$. Calculate the separation of the ships at midday. (Hint: Draw an appropriate diagram.)

4. The crank mechanism shown below has an arm OA of length 30 mm rotating anticlockwise about 0 and a connecting rod AB of length 60 mm. B moves along the horizontal line OB. What is the length OB when OA has rotated by $\dfrac{1}{8}$ of a complete revolution from the horizontal?



Answers

1.

(a) Using the Sine rule $\dfrac{a}{\sin 130^\circ} = \dfrac{6}{\sin 20^\circ} = \dfrac{c}{\sin C}$. From the two left-hand equations $a = 6\dfrac{\sin 130^\circ}{\sin 20^\circ} \simeq 13.44.$

Then, since $C = 30^\circ$, the right hand pair of equations give $c = 6\dfrac{\sin 30^\circ}{\sin 20^\circ} \simeq 8.77.$

(b) Again using the Sine rule $\dfrac{a}{\sin A} = \dfrac{4}{\sin 80^\circ} = \dfrac{3}{\sin C}$ so $\sin C = \dfrac{3}{4}\sin 80^\circ = 0.7386$ there are two possible angles satisfying $\sin C = 0.7386$ or $C = \sin^{-1}(0.7386)$.

These are 47.61° and $180^\circ - 47.614^\circ = 132.39^\circ$. However the obtuse angle value is impossible here because the angle B is 80° and the sum of the angles would then exceed 180° Hence $c = 47.01^\circ$ so $A = 180^\circ - (80^\circ + 47.61^\circ) = 52.39^\circ$.

Then, $\dfrac{a}{\sin 52.39^\circ} = \dfrac{4}{\sin 80^\circ}$ so $a = 4\dfrac{\sin 52.39^\circ}{\sin 80^\circ} \simeq 3.22$

(c) In this case since two sides and the included angle are given we must use the Cosine rule. The appropriate form is

$$b^2 = c^2 + a^2 - 2ca\cos B = 10^2 + 12^2 - (2)(10)(12)\cos 26^\circ = 28.2894$$

so $b = \sqrt{28.2894} = 5.32$

Continuing we use the Cosine rule again to determine say angle C where

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{that is} \qquad 10^2 = 12^2 + (5.32)^2 - 2(1.2)(5.32)\cos C$$

from which $\cos C = 0.5663$ and $C = 55.51^\circ$ (There is no other possibility for C between 0° and 180°. Recall that the cosine of an angle between 90° and 180° is negative.) Finally, $A = 180 - (26^\circ + 55.51^\circ) = 98.49^\circ$.

(d) By the Sine rule

$$\frac{a}{\sin A} = \frac{5}{\sin 50^\circ} = \frac{6}{\sin C} \qquad \therefore \qquad \sin C = 6\frac{\sin 50^\circ}{5} = 0.9193$$

Then $C = \sin^{-1}(0.9193) = 66.82^\circ$ (calculator) or $180^\circ - 66.82^\circ = 113.18^\circ$. In this case both values of C say $C_1 = 66.82^\circ$ and $C_2 = 113.18^\circ$ are possible and there are two possible triangles satisfying the given data. Continued use of the Sine rule produces

(i) with $C_1 = 66.82$ (acute angle triangle) $A = A_1 = 180 - (66.82^\circ + 50^\circ) = 63.18^\circ$ $a = a_1 = 5.83$

(ii) with $C_2 = 113.18^\circ$ $\qquad A = A_2 = 16.82^\circ \qquad a = a_2 = 1.89$

Answers continued

2. We use the Cosine rule firstly to find the angle opposite the longest side. This will tell us whether the triangle contains an obtuse angle. Hence we solve for c using

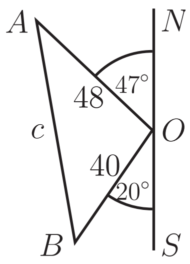
$$c^2 = a^2 + b^2 - 2ab\cos C \qquad 81 = 49 + 36 - 84\cos C$$

from which $84\cos C = 4 \qquad \cos C = 4/84$ giving $C = 87.27°$.

So there is no obtuse angle in this triangle and we can use the Sine rule knowing that there is only one possible triangle fitting the data. (We could continue to use the Cosine rule if we wished of course.) Choosing to find the angle B we have

$$\frac{6}{\sin B} = \frac{9}{\sin 87.27°}$$

from which $\sin B = 0.6659$ giving $B = 41.75°$. (The obtuse case for B is not possible, as explained above.) Finally $A = 180° - (41.75° + 87.27°) = 50.98°$.

3. 

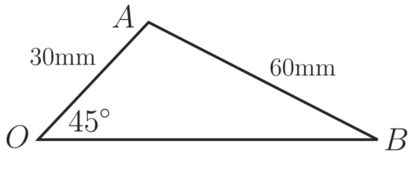
At midday (4 hours travelling) ships A and B are respectively 48 and 40 nautical miles from the port O. In triangle AOB we have

$$AOB = 180° - (47° + 20°) = 113°.$$

We must use the Cosine rule to obtain the required distance apart of the ships. Denoting the distance AB by c, as usual,

$c^2 = 48^2 + 40^2 - 2(48)(40)\cos 113°$ from which $c^2 = 5404.41$ and $c = 73.5$ nautical miles.

4. By the Sine rule $\dfrac{30}{\sin B} = \dfrac{60}{\sin 45} \qquad \therefore \qquad \sin B = \dfrac{30}{60}\sin 45° = 0.353$ so $B = 20.704°$.



(Position after $\frac{1}{8}$ revolution)

The obtuse value of $\sin^{-1}(0.353)$ is impossible. Hence,

$$A = 180° - (45° + 20.704°) = 114.296°.$$

Using the sine rule again $\dfrac{30}{0.353} = \dfrac{OB}{\sin 114.296}$ from which $OB = 77.5$ mm.

Applications of Trigonometry to Waves

4.5

Introduction

Waves and vibrations occur in many contexts. The water waves on the sea and the vibrations of a stringed musical instrument are just two everyday examples. If the vibrations are simple 'to and fro' oscillations they are referred to as 'sinusoidal' which implies that a knowledge of trigonometry, particularly of the sine and cosine functions, is a necessary pre-requisite for dealing with their analysis. In this Section we give a brief introduction to this topic.

Prerequisites

Before starting this Section you should ...

- have a knowledge of the basics of trigonometry
- be aware of the standard trigonometric identities

Learning Outcomes

On completion you should be able to ...

- use simple trigonometric functions to describe waves
- combine two waves of the same frequency as a single wave in amplitude-phase form

1. Applications of trigonometry to waves

Two-dimensional motion

Suppose that a wheel of radius R is rotating anticlockwise as shown in Figure 38.

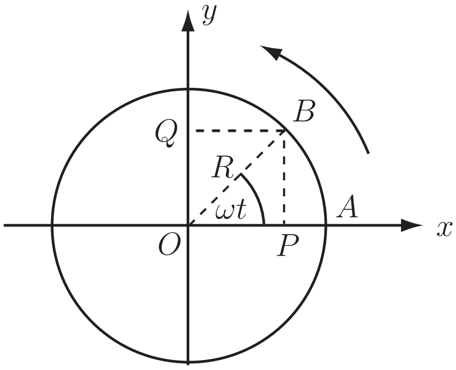


Figure 38

Assume that the wheel is rotating with an angular velocity ω radians per second about O so that, in a time t seconds, a point (x, y) initially at position A on the rim of the wheel moves to a position B such that angle $AOB = \omega t$ radians.
Then the coordinates (x, y) of B are given by

$$x = OP = R\cos\omega t$$

$$y = OQ = PB = R\sin\omega t$$

We know that both the standard sine and cosine functions have period 2π. Since the angular velocity is ω radians per second the wheel will make one complete revolution in $\dfrac{2\pi}{\omega}$ seconds.

The time $\dfrac{2\pi}{\omega}$ (measured in seconds in this case) for one complete revolution is called the **period** of rotation of the wheel. The number of complete revolutions per second is thus $\dfrac{1}{T} = f$ say which is called the **frequency** of revolution. Clearly $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ relates the three quantities introduced here. The angular velocity $\omega = 2\pi f$ is sometimes called the **angular frequency**.

One-dimensional motion

The situation we have just outlined is two-dimensional motion. More simply we might consider one-dimensional motion.

An example is the motion of the projection onto the x-axis of a point B which moves with uniform angular velocity ω round a circle of radius R (see Figure 39). As B moves round, its projection P moves to and fro across the diameter of the circle.

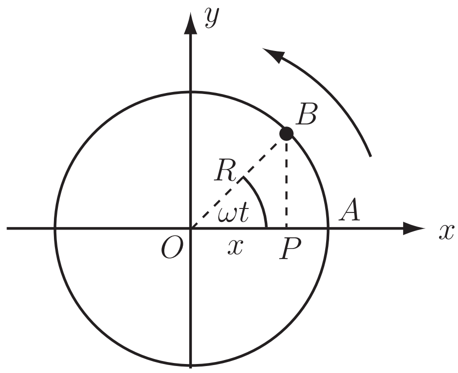


Figure 39

The position of P is given by

$$x = R\cos\omega t \qquad (1)$$

Clearly, from the known properties of the cosine function, we can deduce the following:

1. x varies periodically with t with period $T = \dfrac{2\pi}{\omega}$.
2. x will have maximum value $+R$ and minimum value $-R$.
   (This quantity R is called the **amplitude** of the motion.)

Task

Using (1) write down the values of x at the following times:

$t = 0$, $t = \dfrac{\pi}{2\omega}$, $t = \dfrac{\pi}{\omega}$, $t = \dfrac{3\pi}{2\omega}$, $t = \dfrac{2\pi}{\omega}$.

Your solution

t	0	$\frac{\pi}{2\omega}$	$\frac{\pi}{\omega}$	$\frac{3\pi}{2\omega}$	$\frac{2\pi}{\omega}$
x					

Answer

t	0	$\frac{\pi}{2\omega}$	$\frac{\pi}{\omega}$	$\frac{3\pi}{2\omega}$	$\frac{2\pi}{\omega}$
x	R	0	$-R$	0	R

Using (1) this 'to and fro' or 'vibrational' or 'oscillatory' motion between R and $-R$ continues indefinitely. The technical name for this motion is **simple harmonic**. To a good approximation it is the motion exhibited (i) by the end of a pendulum pulled through a small angle and then released (ii) by the end of a hanging spring pulled down and then released. See Figure 40 (in these cases damping of the pendulum or spring is ignored).

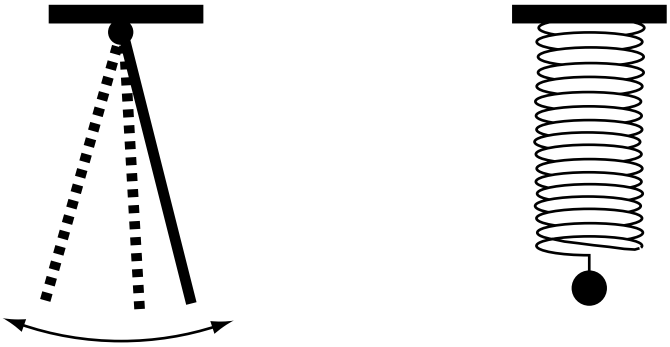


Figure 40

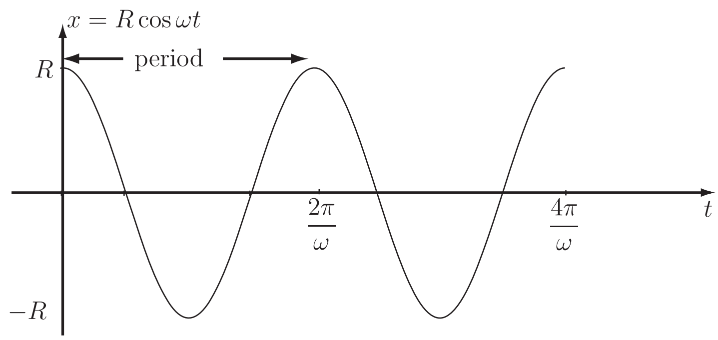
Task

Using your knowledge of the cosine function and the results of the previous Task sketch the graph of x against t where

$$x = R\cos\omega t \quad \text{for } t = 0 \text{ to } t = \frac{4\pi}{\omega}$$

Your solution

Answer

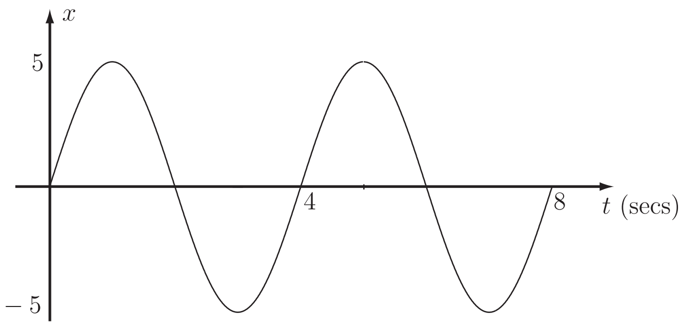


This graph shows part of a cosine **wave**, specifically two **periods** of oscillation. The shape of the graph suggests that the term wave is indeed an appropriate description.

We know that the **shape** of the cosine graph and the sine graph are identical but offset by $\frac{\pi}{2}$ radians horizontally. Bearing this in mind, attempt the following Task.

Task

Write the equation of the wave $x(t)$, part of which is shown in the following graph. You will need to find the period T and angular frequency ω.



Your solution

Answer

From the shape of the graph we have a **sine** wave rather than a cosine wave. The amplitude is 5. The period $T = 4s$ so the angular frequency $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$. Hence $x = 5\sin\left(\frac{\pi t}{2}\right)$.

The quantity x, a function of t, is referred to as the **displacement** of the wave.

Time shifts between waves

We recall that $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ which means that the graph of $x = \sin\theta$ is the same shape as that of $x = \cos\theta$ but is shifted to the right by $\frac{\pi}{2}$ radians.

Suppose now that we consider the waves

$$x_1 = R\cos 2t \qquad\qquad x_2 = R\sin 2t$$

Both have amplitude R, angular frequency $\omega = 2$ rad s^{-1}. Also

$$x_2 = R\cos\left(2t - \frac{\pi}{2}\right) = R\cos\left[2\left(t - \frac{\pi}{4}\right)\right]$$

The graphs of x_1 against t and of x_2 against t are said to have a **time shift** of $\frac{\pi}{4}$. Specifically x_1 is ahead of, or 'leads' x_2 by a time $\frac{\pi}{4}$ s.

More generally, consider the following two sine waves of the same amplitude and frequency:

$$x_1(t) = R\sin\omega t$$

$$x_2(t) = R\sin(\omega t - \alpha)$$

Now $x_1\left(t - \dfrac{\alpha}{\omega}\right) = R\sin\left[\omega\left(t - \dfrac{\alpha}{\omega}\right)\right] = R\sin(\omega t - \alpha) = x_2(t)$

so it is clear that the waves x_1 and x_2 are shifted in time by $\dfrac{\alpha}{\omega}$. Specifically x_1 leads x_2 by $\dfrac{\alpha}{\omega}$ (if $\alpha\ > 0$).

Task

Calculate the time shift between the waves

$$\begin{aligned} x_1 &= 3\cos(10\pi t) \\ x_2 &= 3\cos\left(10\pi t + \frac{\pi}{4}\right) \end{aligned}$$

where the time t is in seconds.

Your solution

Answer

Note firstly that the waves have the same amplitude 3 and angular frequency 10π (corresponding to a common period $\dfrac{2\pi}{10\pi} = \dfrac{1}{5}$ s)

Now $\cos\left(10\pi t + \dfrac{\pi}{4}\right) = \cos\left(10\pi\left(t + \dfrac{1}{40}\right)\right)$

so $x_1\left(t + \dfrac{1}{40}\right) = x_2(t)$.

In other words the time shift is $\dfrac{1}{40}$ s, the wave x_2 **leads** the wave x_1 by this amount. Alternatively we could say that x_1 **lags** x_2 by $\dfrac{1}{40}$ s.

Key Point 20

The equations

$$x = R\cos\omega t \qquad\qquad x = R\sin\omega t$$

both represent waves of amplitude R and period $\dfrac{2\pi}{\omega}$.

The **time shift** between these waves is $\dfrac{\pi}{2\omega}$ because $\cos\left\{\omega\left(t - \dfrac{\pi}{2\omega}\right)\right\} = \sin\omega t$.

The **phase difference** between these waves is said to be $\dfrac{\pi}{2}$ because $\cos\left(\omega t - \dfrac{\pi}{2}\right) = \sin\omega t$

Combining two wave equations

A situation that arises in some applications is the need to combine two trigonometric terms such as

$A\cos\theta + B\sin\theta$ where A and B are constants.

For example this sort of situation might arise if we wish to combine two waves of the same frequency but not necessarily the same amplitude or phase. In particular we wish to be able to deal with an expression of the form

$$R_1\cos\omega t + R_2\sin\omega t$$

where the individual waves have, as we have seen, a time shift of $\dfrac{\pi}{2\omega}$ or a phase difference of $\dfrac{\pi}{2}$.

General Theory

Consider an expression $A\cos\theta + B\sin\theta$. We seek to transform this into the single form $C\cos(\theta - \alpha)$ (or $C\sin(\theta - \alpha)$), where C and α have to be determined. The problem is easily solved with the aid of trigonometric identities.

We know that

$$C\cos(\theta - \alpha) \equiv C(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

Hence if $A\cos\theta + B\sin\theta = C\cos(\theta - \alpha)$ then

$$A\cos\theta + B\sin\theta = (C\cos\alpha)\cos\theta + (C\sin\alpha)\sin\theta$$

For this to be an identity (true for all values of θ) we must be able to equate the coefficients of $\cos\theta$ and $\sin\theta$ on each side.

Hence

$$A = C\cos\alpha \qquad \text{and} \qquad B = C\sin\alpha \qquad (2)$$

Task

By squaring and adding the Equations (2), obtain C in terms of A and B.

Your solution

Answer

$A = C\cos\alpha$ and $B = C\sin\alpha$ gives

$$A^2 + B^2 = C^2\cos^2\alpha + C^2\sin^2\alpha = C^2(\cos^2\alpha + \sin^2\alpha) = C^2$$

$\therefore \quad C = \sqrt{A^2 + B^2}$ (We take the positive square root.)

Task

By eliminating C from Equations (2) and using the result of the previous Task, obtain α in terms of A and B.

Your solution

Answer

By division, $\dfrac{B}{A} = \dfrac{C\sin\alpha}{C\cos\alpha} = \tan\alpha$ so α is obtained by solving $\tan\alpha = \dfrac{B}{A}$. However, care must be taken to obtain the correct quadrant for α.

Key Point 21

If $A\cos\theta + B\sin\theta = C\cos(\theta - \alpha)$ then $C = \sqrt{A^2 + B^2}$ and $\tan\alpha = \dfrac{B}{A}$.

Note that the following cases arise for the location of α:

1. $A > 0,\ B > 0:$ 1st quadrant
2. $A < 0,\ B > 0:$ 2nd quadrant
3. $A < 0,\ B < 0:$ 3rd quadrant
4. $A > 0,\ B < 0:$ 4th quadrant

In terms of waves, using Key Point 21 we have

$$R_1 \cos \omega t + R_2 \sin \omega t = R \cos(\omega t - \alpha)$$

where $R = \sqrt{R_1^2 + R_2^2}$ and $\tan \alpha = \dfrac{R_2}{R_1}$.

The form $R \cos(\omega t - \alpha)$ is said to be the **amplitude/phase** form of the wave.

Example 5

Express in the form $C \cos(\theta - \alpha)$ each of the following:

(a) $3 \cos \theta + 3 \sin \theta$

(b) $-3 \cos \theta + 3 \sin \theta$

(c) $-3 \cos \theta - 3 \sin \theta$

(d) $3 \cos \theta - 3 \sin \theta$

Solution

In each case $C = \sqrt{A^2 + B^2} = \sqrt{9+9} = \sqrt{18}$

(a) $\tan \alpha = \dfrac{B}{A} = \dfrac{3}{3} = 1$ gives $\alpha = 45°$ (A and B are both positive so the first quadrant is the correct one.) Hence $3 \cos \theta + 2 \sin \theta = \sqrt{18} \cos(\theta - 45°) = \sqrt{18} \cos\left(\theta - \dfrac{\pi}{4}\right)$

(b) The angle α must be in the second quadrant as $A = -3 < 0$, $B = +3 > 0$. By calculator : $\tan \alpha = -1$ gives $\alpha = -45°$ but this is in the 4th quadrant. Remembering that $\tan \alpha$ has period π or $180°$ we must therefore add $180°$ to the calculator value to obtain the correct α value of $135°$. Hence

$$-3 \cos \theta + 3 \sin \theta = \sqrt{18} \cos(\theta - 135°)$$

(c) Here $A = -3$, $B = -3$ so α must be in the 3rd quadrant. $\tan \alpha = \dfrac{-3}{-3} = 1$ giving $\alpha = 45°$ by calculator. Hence adding $180°$ to this tells us that

$$-3 \cos \theta - 3 \sin \theta = \sqrt{18} \cos(\theta - 225°)$$

(d) Here $A = 3$ $B = -3$ so α is in the 4th quadrant. $\tan \alpha = -1$ gives us (correctly) $\alpha = -45°$ so

$$3 \cos \theta - 3 \sin \theta = \sqrt{18} \cos(\theta + 45°).$$

Note that in the amplitude/phase form the angle may be expressed in degrees or radians.

Task

Write the wave form $x = 3\cos\omega t + 4\sin\omega t$ in amplitude/phase form. Express the phase in radians to 3 d.p..

Your solution

Answer

We have $x = R\cos(\omega t - \alpha)$ where $R = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3}$ from which, using the calculator in radian mode, $\alpha = 0.927$ radians. This is in the first quadrant $\left(0 < \alpha < \dfrac{\pi}{2}\right)$ which is correct since $A = 3$ and $B = 4$ are both positive. Hence $x = 5\cos(\omega t - 0.927)$.

Exercises

1. Write down the amplitude and the period of $y = \frac{5}{2}\sin 2\pi t$.

2. Write down the amplitude, frequency and time shift of

   (a) $y = 3\sin\left(2t - \dfrac{\pi}{3}\right)$ (b) $y = 15\cos\left(5t - \dfrac{3\pi}{2}\right)$

3. The current in an a.c. circuit is $i(t) = 30\sin 120\pi t$ amp where t is measured in seconds. What is the maximum current and at what times does it occur?

4. The depth y of water at the entrance to a small harbour at time t is $y = a\sin b\left(t - \dfrac{\pi}{2}\right) + k$ where k is the average depth. If the tidal period is 12 hours, the depths at high tide and low tide are 18 metres and 6 metres respectively, obtain $a,\ b,\ k$ and sketch two cycles of the graph of y.

5. The Fahrenheit temperature at a certain location over 1 complete day is modelled by

   $$F(t) = 60 + 10\sin\frac{\pi}{12}(t-8) \qquad 0 \le t \le 24$$

   where t is in the time in hours after midnight.

   (a) What are the temperatures at 8.00 am and 12.00 noon?
   (b) At what time is the temperature $60°$F?
   (c) Obtain the maximum and minimum temperatures and the times at which they occur.

6. In each of the following write down expressions for time-shifted sine and time-shifted cosine functions that satisfy the given conditions:

   (a) Amplitude 3, Period $\dfrac{2\pi}{3}$, Time shift $\dfrac{\pi}{3}$
   (b) Amplitude 0.7, Period 0.5, Time shift 4.

7. Write the a.c. current $i = 3\cos 5t + 4\sin 5t$ in the form $i = C\cos(5\pi - \alpha)$.

8. Show that if $A\cos\omega t + B\sin\omega t = C\sin(\omega t + \alpha)$ then

   $$C = \sqrt{A^2+B^2}, \qquad \cos\alpha = \frac{B}{C}, \qquad \sin\alpha = \frac{A}{C}.$$

9. Using Exercise 8 express the following in the amplitude/phase form $C\sin(\omega t + \alpha)$

   (a) $y = -\sqrt{3}\sin 2t + \cos 2t$ (b) $y = \cos 2t + \sqrt{3}\sin 2t$

10. The motion of a weight on a spring is given by $y = \dfrac{2}{3}\cos 8t - \dfrac{1}{6}\sin 8t$.

    Obtain C and α such that $y = C\sin(8t + \alpha)$

11. Show that for the two a.c. currents

    $$i_1 = \sin\left(\omega t + \frac{\pi}{3}\right) \quad \text{and} \quad i_2 = 3\cos\left(\omega t - \frac{\pi}{6}\right) \text{ then } i_1 + i_2 = 4\cos\left(\omega t - \frac{\pi}{6}\right).$$

12. Show that the power $P = \dfrac{v^2}{R}$ in an electrical circuit where $v = V_0 \cos\left(\omega t + \frac{\pi}{4}\right)$ is

$$P = \frac{V_0^2}{2R}(1 - \sin 2\omega t)$$

13. Show that the product of the two signals

$f_1(t) = A_1 \sin \omega t \qquad f_2(t) = A_2 \sin\{\omega(t + \tau) + \phi\}$ is given by

$$f_1(t)f_2(t) = \frac{A_1 A_2}{2}\{\cos(\omega\tau + \phi) - \cos(2\omega t + \omega\tau + \phi)\}.$$

Answers

1. $y = \dfrac{5}{2}\sin 2\pi t$ has amplitude $\dfrac{5}{2}$. The period is $\dfrac{2\pi}{2\pi} = 1$.

   Check: $y(t+1) = \dfrac{5}{2}\sin(2\pi(t+1)) = \dfrac{5}{2}\sin(2\pi t + 2\pi) = \dfrac{5}{2}\sin 2\pi t = y(t)$

2. (a) Amplitude 3, Period $\dfrac{2\pi}{2} = \pi$. Writing $y = 3\sin 2\left(t - \dfrac{\pi}{6}\right)$ we see that there is a time shift of $\dfrac{\pi}{6}$ in this wave compared with $y = 3\sin 2t$.

   (b) Amplitude 15, Period $\dfrac{2\pi}{5}$. Clearly $y = 15\cos 5\left(t - \dfrac{3\pi}{10}\right)$ so there is a time shift of $\dfrac{3\pi}{10}$ compared with $y = 15\cos 5t$.

3. Maximum current $= 30$ amps at a time t such that $120\pi t = \dfrac{\pi}{2}$. i.e. $t = \dfrac{1}{240}\ s$.

   This maximum will occur again at $\left(\dfrac{1}{240} + \dfrac{n}{60}\right) s, \quad n = 1, 2, 3, \ldots$

4. $y = a\sin\left\{b\left(t - \dfrac{\pi}{2}\right)\right\} + h$. The period is $\dfrac{2\pi}{b} = 12$ hr $\quad \therefore\ b = \dfrac{\pi}{6}$ hr^{-1}.

   Also since $y_{\max} = a + k \quad y_{\min} = -a + k$ we have $a + k = 18 \quad -a + k = 6$ so $k = 12$ m, $a = 6$ m. i.e. $y = 6\sin\left\{\dfrac{\pi}{6}\left(t - \dfrac{\pi}{2}\right)\right\} + 12$.

5. $F(t) = 60 + 10\sin\dfrac{\pi}{12}(t - 8) \qquad 0 \le t < 24$

   (a) At $t = 8:$ temp $= 60°$F. At $t = 12$: temp $= 60 + 10\sin\dfrac{\pi}{3} = 68.7°$F

   (b) $F(t) = 60$ when $\dfrac{\pi}{12}(t-8) = 0,\ \pi,\ 2\pi, \ldots$ giving $t - 8 = 0, 12, 24, \ldots$ hours so $t = 8, 20, 32, \ldots$ hours i.e. in 1 day at $t = 8$ (8.00 am) and $t = 20$ (8.00 pm)

   (c) Maximum temperature is $70°$ F when $\dfrac{\pi}{12}(t = 8) = \dfrac{\pi}{2}$ i.e. at $t = 14$ (2.00 pm).

   Minimum temperature is $50°$F when $\dfrac{\pi}{12}(t - 8) = \dfrac{3\pi}{2}$ i.e. at $t = 26$ (2.00 am).

Answers

6. (a) $y = 3\sin(3t-\pi)$ $\quad y = 3\cos(3t-\pi)$ (b) $y = 0.7\sin(4\pi t-16\pi)$ $\quad y = 0.7\cos(4\pi t-16\pi)$

7. $C = \sqrt{3^2+4^2} = 5$ $\tan\alpha = \dfrac{4}{3}$ and α must be in the first quadrant (since $A = 3,\ B = 4$ are both positive.) $\quad\therefore\ \alpha = \tan^{-1}\dfrac{4}{3} = 0.9273$ rad $\quad\therefore\ i = 5\cos(5t-0.9273)$

8. Since $\sin(\omega t+\alpha) = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$ then $A = C\sin\alpha$ (coefficients of $\cos\omega t$)
$B = C\cos\alpha$ (coefficients of $\sin\omega t$) from which $C^2 = A^2+B^2,\ \sin\alpha = \dfrac{A}{C},\ \ \cos\alpha = \dfrac{B}{C}$

9. (a) $C = \sqrt{3+1} = 2;\ \ \cos\alpha = -\dfrac{\sqrt{3}}{2}$ $\quad \sin\alpha - \dfrac{1}{2}$ so α is in the second quadrant, $\alpha = \dfrac{5\pi}{6}\ \therefore\ \ y = 2\sin\left(2t+\dfrac{5\pi}{6}\right)$ (b) $y = 2\sin\left(2t+\dfrac{\pi}{6}\right)$

10. $C^2 = \dfrac{4}{9}+\dfrac{1}{36} = \dfrac{17}{36}$ so $C = \dfrac{\sqrt{17}}{6}$ $\quad \cos\alpha = \dfrac{-\frac{1}{6}}{\frac{\sqrt{17}}{6}} = -\dfrac{1}{\sqrt{17}}$ $\quad \sin\alpha = \dfrac{\frac{2}{3}}{\frac{\sqrt{17}}{6}} = \dfrac{4}{\sqrt{17}}$

so α is in the second quadrant. $\alpha = 1.8158$ radians.

11. Since $\sin x = \cos\left(x-\dfrac{\pi}{2}\right)$ $\sin\left(\omega t+\dfrac{\pi}{3}\right) = \cos\left(\omega t+\dfrac{\pi}{3}-\dfrac{\pi}{2}\right) = \cos\left(\omega t-\dfrac{\pi}{6}\right)$

$\therefore\ \ i_1+i_2 = \cos\left(\omega t-\dfrac{\pi}{6}\right)+3\cos\left(\omega t-\dfrac{\pi}{6}\right) = 4\cos\left(\omega t-\dfrac{\pi}{6}\right)$

12. $v = V_0\cos\left(\omega t+\dfrac{\pi}{4}\right) = V_0\left(\cos\omega t\cos\frac{\pi}{4}-\sin\omega t\sin\frac{\pi}{4}\right) = \frac{V_0}{\sqrt{2}}\left(\cos\omega t-\sin\omega t\right)$

$\therefore\quad v^2 = \dfrac{V_0^2}{2}(\cos^2\omega t+\sin^2\omega t-2\sin\omega t\cos\omega t) = \dfrac{V_0^2}{2}(1-\sin 2\omega t)$

and hence $\quad P = \dfrac{v^2}{R} = \dfrac{V_0^2}{2R}(1-\sin 2\omega t.)$

13. Since the required answer involves the difference of two cosine functions we use the identity

$$\cos A-\cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

Hence with $\dfrac{A+B}{2} = \omega t,\ \ \dfrac{B-A}{2} - \omega t+\omega\tau+\phi.$

We find, by adding these equations $B = 2\omega t+\omega\tau+\phi$ and by subtracting $A = -\omega\tau-\phi.$

Hence $\quad \sin(\omega t)\sin(\omega t+\omega\tau+\phi) = \dfrac{1}{2}\{\cos(\omega\tau+\phi)-\cos(2\omega t+\omega\tau+\phi)\}.$

(Recall that $\cos(-x) = \cos x.$) The required result then follows immediately.